Equilibria in the CAPM with non-tradeable endowments

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ARTICLE INFO

Article history:
Received 27 November 2015
Received in revised form 19 October 2017
Accepted 18 December 2017
Available online 31 January 2018

Keywords:
Portfolio choice
CAPM
Non-tradeable endowments
Risk aversion
Equilibrium
Incomplete markets

ABSTRACT

This paper establishes existence and uniqueness of equilibria in the capital asset pricing model (CAPM) in a setting with incomplete markets in which part of the endowments are non-tradeable. It is shown that in equilibrium, agents are willing to assume aggregate hedgeable risk of the market but will no longer hold fractions of the market portfolio. The paper studies the effects of non-traded endowments on equilibrium asset prices and allocations and establishes a linear pricing formula, a security market line, and conditions for the positivity of asset prices.

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1. Introduction

Despite its well-known limitations, the Capital Asset Pricing Model (CAPM) continues to be one of the most widely used equilibrium models in practice. Its applications range from serving as a theoretical basis for investment advice to providing valuation formulas for the assessment of new ventures. The discussion among researchers about its theoretical consistency and its empirical validity has persisted to this day. Recent contributions to the ongoing debate include, among others, Fama and French (2004), Markowitz (2005), Levy (2007), and Levy (2011). The classical equilibrium setting of the CAPM is that of a financial market with finitely many assets in which agents’ endowments are fully tradeable. The cornerstones of the theory are results on the existence of equilibrium prices and allocations and their properties. One of its central results states that in equilibrium, agents hold a combination of the market portfolio and the risk-free asset. Moreover, equilibrium asset prices lead to a linear pricing formula, which is widely used to value the payoffs of new projects. Yet, in reality and in contrast to the classic assumptions, not all of an agent’s endowment may be tradeable. For example, an endowment may fail to be trade because it includes a privatively owned firm or because it is exposed to sources of risk other than market price movements. Given the ubiquity of the CAPM in applications, it is therefore important to understand not only the extent to which investment advice and pricing rules remain valid when non-tradeable endowments are taken into account, but also the conditions under which equilibrium prices and allocations exist.

The literature on the CAPM with non-tradeable endowments so far has investigated properties of equilibrium prices and allocations assuming equilibria exist, e.g., see Mayers (1972, 1976) and Oh (1996). Mayers (1972) shows that agents no longer hold a portion of the market portfolio when a part of the endowments is non-tradeable and that risk-premia change significantly. Oh (1996) studies the effects of non-marketable assets on prices. His assumptions on preferences and probability distributions rule out wealth effects and thus are rather restrictive. However, to ensure that the model is suitable and the analysis is relevant, it is necessary to establish existence of equilibria in the first place. The literature on the existence of equilibria in the CAPM is extensive, but applies exclusively to the case of tradeable endowments, e.g., see Nielsen (1988, 1990a, b), Allingham (1991), Dana (1999), Hens et al. (2002), Won et al. (2008), or Wenzelburger (2010). It is far from being obvious how these results can be extended to include non-tradeable endowments. With the exception of unpublished work by Hara (2001), no existence results for non-tradeable endowments have been established. Hara (2001) considers incomplete markets and allows for non-tradeable individual endowments, but all of his results rely on the assumption that aggregate endowment is tradeable.

* We would like to thank Stefan Reimann for numerous valuable discussions and Gevorg Hunanyan, Sandro Flury, and Conrad Spanaus for important comments. We are very grateful to an anonymous referee and an associate editor for insightful comments and suggestions.
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https://doi.org/10.1016/j.jmateco.2017.12.004
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The objective of this paper is therefore twofold. The first one is to establish the missing existence results for the CAPM with endowments that may contain a non-tradeable component and an aggregate endowment that may lie outside the marketed space. The second one is to extend the well-known classical results on equilibrium prices and allocations to this more general setting. We start from a traditional CAPM in which agents are characterised by general mean–variance preferences. In contrast to the bulk of the literature, we assume that agents’ endowments consist of a tradeable and a non-tradeable component. The tradeable endowment consists of a portfolio of marketed assets. The non-tradeable endowment of an agent may be thought of as a privately owned firm whose ownership rights are not marketed but whose profits may be correlated to market prices. Thus, contrary to the literature (LeRoy and Werner, 2014), the non-tradeable endowment need not be orthogonal to the marketed space but may still be partially hedgeable with marketed assets. In addition to its decomposition into a tradeable and a non-tradeable component, an agent’s endowment therefore admits a second natural decomposition into a hedgeable and a non-hedgeable component, which is orthogonal to the marketed subspace. This decomposition turns out to be essential for our findings and highlights the differences isorthogonal to the marketed subspace. This decomposition turns out to be essential for our findings and highlights the differences between the extended and the classical CAPM.

The centre piece of our analysis is a generalised Two-fund Separation Theorem. Its classical form was formulated for asset-demand functions by Lintner (1965) and states that for any given level of prices, the optimal portfolio is a combination of the risk free asset and a mutual fund. It turns out that in equilibrium, this mutual fund is the market portfolio. When agents are endowed with non-tradeable endowments, we show that a two-fund separation theorem still holds. However, an agent’s optimal exposure to risk now depends on the non-hedgeable component of her endowment and her optimal portfolio on the hedgeable part of her non-tradeable endowment. Existence of CAPM equilibria then follows from the Two-fund Separation Theorem, which is used to reduce a multivariate existence problem in the asset market to a uni-variate existence problem in the market for hedgeable risk. This reduction is used to show that CAPM equilibria exist.

Based on the existence result, a number of analogies as well as differences to the classical CAPM are deduced. First, we highlight that in equilibrium agents will no longer invest into the same mutual fund. The hedgeable component of an optimal consumption stream now corresponds to a combination of the risk-free asset and the extended market portfolio, i.e., the portfolio that generates the payoff of the hedgeable part of aggregate endowment. Second, we establish that in equilibrium marketed payoffs are priced according to a linear pricing formula, which is similar to the classical pricing formula but differing in that the market payoff is replaced by the payoff of the extended market portfolio. Third, we derive a security market line, establishing a positive relation between risk and expected return. This confirms the classical distinction between diversifiable and non-diversifiable risk. Fourth, we generalise findings by Levy (2007) for the traditional CAPM by providing conditions on the payoff structure of financial assets and agents’ preferences that ensure the positivity of equilibrium asset prices. Finally, we examine the effect of non-hedgeable risk on equilibrium prices. An increase in the non-hedgeable risk of an agent may change the equilibrium price of a traded asset in either direction, depending on the elasticity of the agent’s risk aversion and the correlation of the asset’s payoff with the payoff of the aggregate hedgeable endowment.

The paper is organised as follows. Section 2 introduces the underlying model assumptions. Our Two-fund Separation Theorem is established in Section 3. In Section 4, we prove the existence of CAPM-equilibria in the presence of non-tradeable endowments and provide an elementary condition for their uniqueness. Properties of CAPM-equilibria are studied in Section 5. All technical proofs are relegated to an Appendix which also contains a technical analysis of risk-taking behaviour.

### 2. The model

Our setup is standard for general equilibrium in incomplete markets. We consider a one-period economy with dates 0 and 1. Uncertainty at time 1 is modelled by a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega\) describes the possible states of nature, \(\mathcal{F}\) the set of observable events, and \(\mathbb{P}\) their probability distribution. There is only one consumption good and consumption takes place at date 1 only. The consumption space \(\mathcal{C}\) is a subspace of \(\mathbb{C}^2\), the Hilbert space of all square-integrable random variables on \((\Omega, \mathcal{F}, \mathbb{P})\), which is endowed with the inner product \(\langle c|e \rangle := \mathbb{E}[c|e] \) for \(c, e \in \mathbb{C}^2\). For a typical random variable \(c \in \mathcal{C}\), we denote by \(\mathbb{E}[c], \text{Var}[c]\) and \(\sigma[c]\) the expected value, the variance, and the standard deviation, respectively. The covariance between two random variables \(c, e \in \mathcal{C}\) is denoted by \(\text{Cov}[c, e]\).

#### Financial market.

The financial market consists of \(K\) risky assets with non-zero positive payoffs \(q_1, \ldots, q_K \in \mathbb{C}\) and prices \(p_1, \ldots, p_K \in \mathbb{R}\) as well as a risk-free asset with deterministic payoff \(R\) whose price is normalised to 1. The set \(\{q_1, \ldots, q_K\}\) is assumed to be linearly independent. Payoffs and prices of the risky assets are summarised in the \(K\)-tuples \(q := (q_1, \ldots, q_K)\) and \(p := (p_1, \ldots, p_K)\), respectively. Their expected payoffs are denoted by \(\mathbb{E}[q] := (\mathbb{E}[q_1], \ldots, \mathbb{E}[q_K])\) so that the vector \(\pi := \mathbb{E}[q] - R\mathbb{E}[p] \in \mathbb{R}^K\) represents expected excess returns. We denote the covariance matrix of the risky payoffs by \(V := (V_{ij})\), where

\[
V_{ij} := \text{Cov}[q_i, q_j]
\]

is the covariance between the payoff of the \(k\)th and the \(l\)th risky asset. Since the risk-free asset is not a linear combination of risky assets, the covariance matrix \(V\) is symmetric and positive definite and hence invertible.

Portfolios of financial assets are described by vectors \((x, a) \in \mathbb{R}^K \times \mathbb{R}\), where \(x_k\) denotes the number of units of the \(k\)th risky asset and \(a\) the number of units of the risk-free asset. Denoting the standard Euclidean product on \(\mathbb{R}^K\) by \((\cdot, \cdot)\), the payoff of the portfolio \((x, a)\) is \((q, x) + R\mathbb{E}[p]\) and its date-0 value is \((p, x) + a\). Risky assets are in positive net supply, which is denoted by \(x_n \in \mathbb{R}^K_\downarrow\) and referred to as market portfolio. The risk-free asset is in zero net supply and there are no short-sale constraints in the market. The payoffs of the financial assets span the marketed subspace

\[
\mathcal{M} := \text{span}\{R_1, q_1, \ldots, q_K\}
\]

of the consumption space \(\mathcal{C}\), which induces the orthogonal decomposition of \(\mathcal{C}\) into

\[
\mathcal{C} = \mathcal{M} \oplus \mathcal{M}^\perp.
\]

Since the payoff \(R\) of the risk-free asset is contained in \(\mathcal{M}\), we have \(\mathbb{E}[e] = 0\) for all \(e \in \mathcal{M}^\perp\) and the inner product of any random variable in \(\mathcal{C}\) with a random variable in \(\mathcal{M}^\perp\) coincides with its covariance, that is, \(\langle c|e \rangle = \text{Cov}[c, e]\) for all \(c \in \mathcal{C}, e \in \mathcal{M}^\perp\).

#### Individual endowments.

There is a finite number \(I\) of agents. Each agent \(i\) is endowed with a date-1 consumption pattern \(e_i \in \mathcal{C}\), consisting of the payoff of a portfolio of risky assets \(x_{0i} \in \mathbb{R}^K\) and a non-tradeable payoff \(n_i \in \mathcal{C}\), such that

\[
e_i = (q, x_{0i}) + n_i.
\]

In contrast to most of the literature, we do not assume that \(n_i\) is orthogonal to the marketed subspace \(\mathcal{M}\), so that \(n_i\) may be partially hedged by a portfolio of traded assets. In particular, there exists a unique portfolio of assets \((y_{0i}, b_{0i})\) and a unique payoff \(e_{ni} \in \mathcal{M}^\perp\) such that

\[
n_i = (q, y_{0i}) + R_j b_{0i} + e_{ni} \in \mathcal{M} \oplus \mathcal{M}^\perp.
\]
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