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Derivatives pricing with market impact and limit order book ☆

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ABSTRACT

This paper investigates the derivatives pricing under the existence of liquidity costs and market impact for the underlying asset in continuous time. First, we formulate the charge for the liquidity costs and the market impact on the derivatives prices through a stochastic control problem that aims to maximize the mark-to-market value of the portfolio less the quadratic variation multiplied by a risk aversion parameter during the hedging period and the liquidation cost at maturity. Then, we obtain the derivatives price by reduction of this charge from the premium in the Bachelier model. Second, we consider a second order semilinear partial differential equation (PDE) of parabolic type associated with the control problem, which is analytically solved or approximated by an asymptotic expansion around a solution to an explicitly solvable nonlinear PDE. Finally, we present the numerical examples of the pricing for a variance option and a European call option, and show comparative static analyses.

1. Introduction

In this paper, we consider the derivatives pricing under the existence of the market impact and the liquidity costs on the underlying asset price, which are caused by the transactions of the hedger. After formulating the hedging cost through a stochastic control problem, which is a generalized form of a linear-quadratic control problem, we provide a scheme to compute the cost which is solved analytically or approximated by an asymptotic expansion of a second order semilinear PDE of parabolic type. This asymptotic expansion is novel in that the solution is expanded around that of an explicitly solvable semilinear PDE. This is different from previous works on asymptotic expansions for derivatives pricing (see Takahashi, 2015 and the references therein), which typically make expansion around linear PDEs.

Estimation of the total liquidity cost during the hedging period is the most essential factor in pricing in practice, since banks may have losses by the price spreads which they pay in every hedging transaction. Prediction of the effect of market impact on the hedging cost is also important, especially when banks trade derivatives on illiquid underlying assets such as low liquidity stocks and illiquid foreign exchange rates. Moreover, when banks quote a derivatives price in biddings, the estimation of these costs is occasionally the only differentiator among the participants. Despite these facts, the estimation of the entire hedging cost is usually done by the traders' rules of thumb. This study provides a quantitative method to estimate the cost. In addition, our model incorporates the liquidation cost at maturity for the two settlement types, physical and cash settlements, and slippages on execution volume of the underlying asset, which are observed in the trading of illiquid assets in practice.

As for related literatures, Li and Almgren (2016) deal with hedging an option under the existence of liquidity costs and market impact. Guéant and Pu (2015) consider indifference pricing of a hedger with an exponential utility on the mark-to-market value of the hedging portfolio at maturity. After deriving the HJB equation for the optimization problem, Guéant and Pu (2015) solve the HJB equation numerically by finding a maximum point at every grid of the discretized equation.

On the other hand, we adopt a different objective function from the one in Guéant and Pu (2015) for the maximization. By assuming the mark-to-market value of the hedging portfolio at maturity less the terminal liquidation cost and the quadratic hedging error as the objective function to be maximized, the problem becomes a generalized form of the linear-quadratic control problem, where the related HJB equation reduces to a second order semilinear PDE of parabolic type. Then, depending on the payoff of derivatives, we analytically solve the semilinear PDE or asymptotically expand the solution of the PDE up to the first order. In detail, we expand the solution around that of a solvable semilinear PDE. The zeroth order part of the solution has a quadratic expression with respect to a state variable, whose coefficients satisfy an ODE or a linear PDE, and the first order part is a solution of a second order linear

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PDE. We solve the system of the ODE and PDEs through stochastic representations of the solutions by the Feynman–Kac formula.

We note that Li and Almgren (2016) only deal with intraday hedging on a specific date far away from the maturity, and hence does not consider pricing options, which is generalized in our study to hedging and pricing derivatives for the entire trading period. In particular, we remove the crucial assumption in Li and Almgren (2016) that the derivatives gamma is constant. This assumption is only applicable to intraday trading of a specific trading date far from the maturity. Note that Li and Almgren (2016) deal with the general gamma only in a no market impact case. When the underlying asset price is staying in the at-the-money area and the trading date is near the maturity, calculation of the optimal hedging strategy is particularly important since the hedging is difficult due to large derivatives gamma for European call or put options. Further, in contrast to Li and Almgren (2016), the liquidation cost at maturity for the two settlement types, physical and cash settlements, and slippages on execution volume of the underlying asset are also incorporated in our model.

We remark that the derivatives gamma is the second order differential of the value of the derivatives with respect to the underlying asset price. The delta, the first order differential of the value of the derivatives with respect to the underlying asset price, is the quantity of the underlying asset to offset in order to keep the mark-to-market value of the hedger's portfolio unaffected from underlying asset price movements in a short period. Thus, when the gamma is high, the hedger has to trade a large number of units for the underlying assets every time the underlying asset price moves. The gamma is particularly high when the trading date is near the maturity or the underlying asset price is staying in the area where the convexity of the derivatives payoff is large, such as at-the-money area of the European call or put option.

While Li and Almgren (2016) do not show any numerical experiment and Guéant and Pu (2015) provide only one example with market impact, our study presents various cases of derivatives prices under the existence of market impact. We provide derivatives prices for a variance option in physical settlement, which are analytically solvable, and those for a European call option in physical settlement, which are obtained through the asymptotic expansion. Note that the derivatives with a variance option is an important example corresponding to a variance contract that pays realized variance of the underlying asset price at maturity.

As for numerical methods for solving the HJB equations, Aliyu (2003) investigates a transformation approach for solving the equation by reducing the equation into a set of coupled algebraic-differential inequalities. Huang, Wang, Chen, and Li (2006) propose a semi-meshless discretization where the spatial discretization is based on a collocation scheme using the global radial basis functions. Beard, Saridis, and Wen (1997) apply Galerkin's approximation method to a solution of the generalized HJB equation in a deterministic control problem. Crespo and Sun (2003) present a method for finding optimal controls by the generalized cell mapping and the short-time Gaussian approximation scheme.

This paper is organized as follows: Section 2 explains our model. Section 3 provides an asymptotic expansion of an associated semi-linear PDE with its coefficients' computation in Section 4. Section 5 provides examples with numerical experiments in Section 6. Section 7 compares our derivatives price with that of Guéant and Pu (2015). Section 8 concludes.

2. Model

In this section, we introduce an optimal hedging problem for a derivatives hedger, who is the sole rational large investor in the market of the underlying asset, under the existence of liquidity costs and market impact.

Since estimation of costs related to illiquidity is essential in derivatives pricing in practice, we incorporate important factors on illiquidity (a finite variation process for the units of orders submitted by the hedger, temporary and permanent impacts on prices, execution slippages on the trade units, and the liquidation cost at maturity which depends on the settlement types) in modeling.

First, we assume a finite variation process for units of orders submitted by the hedger in Section 2.1. This is different from the delta hedging in the Black-Sholes and the Bachelier model, since the delta hedging in these models can be done at any instant to offset the fluctuation of the derivatives price, and thus the underlying asset position has an infinite total variation, which is impossible in practice especially for illiquid markets.

Also, we take market frictions into consideration, i.e. slippages on the trade units, temporary and permanent impacts on the mid price, and the liquidity cost at maturity in Section 2.2.

Specifically, the hedger enters a long/short derivatives position and starts hedging the position with the underlying asset. The hedger mark-to-markets the portfolio with the Bachelier model for the derivatives position and with the mid price for the underlying asset position. At inception of the trading, the hedger exchanges the initial delta units of the underlying asset based on the Bachelier model at mid price with the counterparty of the derivatives. We explain these points in detail in the following subsections.

2.1. Order volume and asset price processes

The hedger aims to maximize his/her expected utility which is risk neutral or risk averse. The hedger holds the derivatives position at inception of the trading by paying m_0 as the premium.

Let $[0, T]$ be the trading period, where 0 is the initial time of the trading and T is the maturity of derivatives. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbf{P})$ be the filtered probability space satisfying the usual conditions. We consider an economy that consists of a money market account and an underlying asset. We assume that the risk-free interest rate is zero, which implies that the price of the money market account is always 1. Let θ_t be a $\{\mathcal{F}_t\}$-adapted process which satisfies $\mathbf{E}\left[\int_0^T \theta_s^2 ds\right] < \infty$, $(W_t, W_t^{\perp})$ be a two dimensional $\{\mathcal{F}_t\}$-Brownian Motion, σ and δ be positive constants, and ϵ be nonnegative. Let X_t be the number of units of the submitted orders by the hedger to market makers by time t, which is differentiable with derivative θ_t, that is, $X_t = \int_0^t \theta_s ds$. In other words, θ_t is the number of units of orders for the underlying asset that the hedger submits to the market makers per period, or the speed of order placement for the underlying asset for hedging. (buy orders (when the sign is positive) or sell orders (when the sign is negative)).

The reason why we assume absolute continuity with respect to time for the accumulated order volume from the hedger is that in the real world, the order volume has a finite total variation, which corresponds to the integrability of θ, $\int_0^T |\theta_s| ds < \infty$. If X includes a term of a stochastic integration with respect to a Brownian motion, the total variation of the hedger's position can be infinite in the finite interval. This corresponds to an infinite order submission volume, which is impossible in the real word. Hence, the absolute continuity of the underlying asset position is particularly important in trading of an illiquid asset as in Longstaff (2001), for example.

We define the mid price process of the underlying asset P_t as

$$P_t = P_0 + \sigma W_t + \epsilon X_t,\ 0 \le t \le T. \tag{1}$$

This indicates that the permanent market impact on the mid price process ϵX_t is proportional to X_t, the number of units of the orders submitted by the hedger until time t. This means that if the hedger, the sole large trader, places a large number of buying (selling) orders, then there is a positive (negative) value of the permanent