Technical note: Solving inventory models by algebraic method

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ABSTRACT

We consider the open question raised by Chang et al. (2005) to solve the EOQ and EPQ inventory models without referring to calculus. Lau et al. (2016) and Chiu et al. (2017) both extended this open question by deriving criterion for the existence and uniqueness of the interior minimum solution but they used analytical techniques that are related to calculus. Moreover, their derivations are incomplete and contained questionable results. In this note, we only used algebraic approach for their extended open question.

1. Introduction

Since Grubbström and Erdem (1999) used an algebraic approach to solve the minimum problem of an inventory model, there are nearly two hundred papers that followed this trend to develop different algebraic methods to find the optimal solutions for inventory systems. Most papers concentrated on their own inventory models and did not pay attention to an open question proposed by Chang et al. (2005), which contains a quadratic polynomial inside a square root and requires zero and one partial derivatives of calculus. By Equation (2), they provided the next open minimum problem:

C(Q,B) = cD + h [(1 + β/Q)β² + 2pKD/B - b]

Specifically, Chang et al. (2005) mentioned that an alternative way to solve their minimum problem is to solve the following problem:

C(Q,B) = cD + h [(1 + β/Q)β² + 2pKD/B - b] (3)

By Equation (2), they provided the next open minimum problem:

√(1 + α)β² + β - B

without using partial derivatives of calculus, where α = b/h, β = 2KD/h, α > 0 and β > 0.

Recently, there are two papers: Lau et al. (2016) and Chiu et al. (2017) to consider the following more generalized minimum problem:
\[ f(x) = \sqrt{a^2x^2 + bx + c} - x \]  
(4)

with \( f(x) > 0 \), for \( x > 0 \), to secure the minimum problem has an interior optimal solution.

Lau et al. (2016) obtained two cases:

(a) \( a > 1, b \leq 0, c > 0 \) and \( 4ac > b^2 \), and  
(b) \( a > 1, b > 0, c > 0 \) and \( 4ac > b^2 \).

However, their solution contained questionable results, which will be demonstrated by our Theorem 1 in Section 3, and several of their derivations were derived by calculus.

Chiu et al. (2017) claimed two cases:

(a) \( a > 1, c > 0 \) and \( 4ac > b^2 \), and  
(b) \( a > 1 \) and \( 4c > b^2 \).

They used the knowledge of calculus in derivations and their findings are incomplete, which will be demonstrated by our Theorem 1 as well. Hence, in this note, we will provide further improvements for Lau et al. (2016) and Chiu et al. (2017) with algebraic method.

3. Proposed algebraic method

Our goal is to find conditions to guarantee that \( f(x) = \sqrt{a^2x^2 + bx + c} - x \) for \( x > 0 \) with \( f(x) > 0 \) has a unique minimum (optimal) solution by algebraic methods. Before proving our theorem, we would firstly explain the reason why we concentrate on \( x > 0 \) with \( f(x) > 0 \) herein.

When \( x \rightarrow 0^+ \) and \( f(x) \rightarrow 0 \), it implies that the inferior value occurs on the boundary. Thus, the original inventory model has an inferior value when \( Q=0^+ \), which is violating the common sense of the original minimum cost inventory model. As a Remark, for the original minimum cost inventory model, the average set up cost will go to infinite when the replenishment cycle approaches to zero. Therefore, the original minimum cost inventory model cannot have the finding as \( Q=0^+ \). If the optimal solution of \( f(x) \) satisfies \( x \rightarrow \infty \), it implies that the optimal order quantity as well as the holding cost will go to infinite. Thus, \( x \rightarrow \infty \) is not an acceptable optimal solution for \( f(x) \).

Under the condition of a point, say \( x \), satisfying \( 0 < x < \infty \) and \( f(x) \leq 0 \), the corresponding inventory model will have negative or zero holding cost and shortage cost that is a contradiction for inventory models. Hence, we look for restrictions to guarantee \( x > 0 \) and \( f(x) > 0 \).

Remark. We want to rule out the case of \( a = 1, c > 0 \) and \( b = 2\sqrt{c} \) to result in the trivial case \( f(x) = \sqrt{x^2 + 2\sqrt{c}x + c} - x \equiv \sqrt{c} \) that has infinite minimum solutions.

In this section, we will prove two necessary conditions of \( c > 0 \) and \( a > 1 \), and rewrite Equation (4) to obtain additional three conditions: \( 4ac - b^2 > 0, 4c > b^2 \) when \( b > 0 \), and \( 4(a-1)c > b^2 \) when \( b < 0 \) for deriving our proposed Theorem 1.

First, we will prove that \( c > 0 \). Assuming \( c < 0 \), we select a point \( x_1 \) with \( x_1 > 0 \) and \( ax_1^2 + bx_1 + c < 0 \), which is a contradiction with \( \sqrt{ax_1^2 + bx_1 + c} \).

Thus, we select \( x_1 = \frac{1}{\sqrt{a}} \frac{1}{\sqrt{|c|}} \) with \( s = \min \left\{ 1, \frac{|\sqrt{1+|a|} + \sqrt{|b|}}{2\sqrt{|c|}} \right\} \). As \( s \leq 1 \), we compute

\[
ax_1^2 + bx_1 + c < (1 + |a|)x_1^2 + (1 + |b|)x_1 - |c| = \left(1 + \frac{|b|}{1 + |a|}\right)(s^2 + s) - |c|
\]

\[
= 2s \left(1 + \frac{|b|}{1 + |a|}\right) - |c| \leq 0
\]

(5)
to imply an unacceptable result \( f(x_1) < 0 \). With an assumption of \( c < 0 \), we construct a sequence \( \{x_n\} \) as \( x_n = \frac{1}{n^2} \) and \( t_n = \sqrt{|c_n|} \), and evaluate that

\[
f(t_n) = \sqrt{ax_n^2 + bx_n + c} - t_n \leq \sqrt{|a|}t_n^2 + \sqrt{|b|}t_n + t_n \leq (1 + |a| + |b|)\sqrt{|c_n|}
\]

\[
= \frac{1 + |a| + |b|}{n}
\]
to yield \( f(t_n) \to 0 \). As it will result in a solution of \( x \to 0^+ \), we derive the first condition of

\[
c > 0.
\]

(6)

Second, we will prove that \( a > 1 \). When \( a < 1 \), we take \( x_0 = m \left(\frac{1+b}{1-a}\right) \) with \( m = 1 + \frac{\sqrt{1-4ac}}{1+b} > 1 \) to imply that

\[
(1-a)x_0^2 - (1 + |b|)x_0 = (1-a)m^2 \left(1 + |b|\right) \frac{1}{1-a} - (1 + |b|)m \frac{1 + |b|}{1-a} \]

\[
= \frac{1 + |b|}{1-a}m(m-1) > \frac{1 + |b|}{1-a}(m-1)^2
\]

\[
> \frac{(1 + |b|)^2 (1-a)c}{1 + |b|} = c \geq c.
\]

(7)

From \( (1-a)x_0^2 > (1 + |b|x_0 > (1-b)x_0 + c > bx_0 + c \), it yields \( f(x_0) < 0 \) that is a contradiction, and thus, \( a < 1 \) is not acceptable. The condition of \( a \geq 1 \) is derived. We further show that when \( a = 1 \), a minimum solution for \( 0 < x < \infty \) cannot be found. Note that the inferior value occurs on the boundary \( x = 0 \) or \( x = \infty \) are out of the domain \( 0 < x < \infty \). When \( a = 1 \), three cases are elaborated separately herein: (a) \( b > 2\sqrt{c} \), (b) \( b < 2\sqrt{c} \) and (c) \( b = 2\sqrt{c} \).

For case (a), when \( a = 1 \) and \( b > 2\sqrt{c} \), we derive that

\[
4c < b^2 \iff ax^2 + bx + c > x^2 + 2\sqrt{c}x + c \iff \sqrt{ax^2 + bx + c} > x + \sqrt{c} \iff f(x) > \sqrt{c}.
\]

(8)

Thus, the inferior value occurs when \( x \to 0^+ \).

For case (b), \( a = 1 \) and \( b < 2\sqrt{c} \), we show that \( f(x) > b/2 \) for all \( x \) when

\[
f(x) > b/2 \iff \frac{bx + c}{\sqrt{x^2 + bx + c}} > \frac{b}{2} \iff \sqrt{x^2 + bx + c} > \frac{b}{2} \iff \sqrt{x^2 + bx + c} > \frac{b}{2}(x + c) \iff (bx + c) \left(1 - \frac{b}{4}\right) > 0.
\]

(9)

Next, we consider \( f(x) \) when \( x \to \infty \). Since \( f(x) = \frac{b(1+b)}{\sqrt{1+b^2(x^2+c)+1}} \)

\[
f(x) \to \frac{b(1+b)}{1+b^2} \text{ when } x \to \infty.
\]

Thus, the inferior value happens when \( x \to \infty \).

For case (c), if \( a = 1 \) and \( b = 2\sqrt{c} \), \( f(x) \) is a constant function with \( f(x) \equiv \sqrt{c} \). Thus, every positive point is considered as the optimal solution, which violates our goal of finding a unique minimum solution. Consequently, we derive the second condition

\[
a > 1.
\]

(10)

We rewrite Equation (4) as

\[
\sqrt{ax^2 + bx + c} = x + f(x)
\]

(11)

and take square on both sides. We arrange the expression in the descending order of \( x \) and treat \( f(x) \) as a constant term for the moment. Through the square for \( x \), it implies

\[
\sqrt{ax^2 + bx + c} = x + f(x)
\]

(12)
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