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The Harrison–Pliska arbitrage pricing theorem under transaction costs

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Abstract

We consider a simple multi-asset discrete-time model of a currency market with transaction costs assuming the finite number of states of the nature. Defining two kinds of arbitrage opportunities we study necessary and sufficient conditions for the absence of arbitrage. Our main result is a natural extension of the Harrison–Pliska theorem on asset pricing. We prove also a hedging theorem without supplementary hypotheses.

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1. Introduction

The famous result of Harrison and Pliska (1981), known also as the 'fundamental theorem' on asset (or arbitrage) pricing (FTAP) asserts that a frictionless financial market is free of arbitrage if and only if the price process is a martingale under a probability measure equivalent to the objective one. The original formulation involved the assumption that the underlying probability space $(\Omega, \mathcal{F}, P)$ (in other words, the number of states of the nature) is finite; it has been removed in the subsequent study of Dalang et al. (1990). It is worth to note that the passage from finite to infinite Ω is by no means trivial: instead of purely geometric considerations (which make the Harrison–Pliska theorem so attractive for elementary courses in financial economics) much more delicate topological or measure-theoretical arguments must be used. These mathematical aspects were investigated in details by a number of authors (e.g. Stricker, 1990; Schachermayer, 1992; Kabanov and Kramkov, 1994; Rogers,

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1994; Jacod and Shiryayev, 1998). The aim of this note is to present an extension of the arbitrage pricing theorem for a multi-asset multi-period model with finite Ω and proportional transaction costs. We use the geometric formalism developed in Kabanov (1999), Delbaen et al. (2001) and Kabanov and Last (2001). Our result makes clear that the concept of the equivalent martingale measure, though useful in the context of frictionless market models, has no importance (and even misleading) in a more realistic situation of transaction costs. We track how the dual variables in our more general model "degenerate" into densities of martingale measures. Our paper contains also a hedging theorem which is free from any auxiliary assumptions (cf. with results in Kabanov (1999), Delbaen et al. (2001) and Kabanov and Last (2001)). In spite the results being mathematically simple, they are not deductible, to the best of our knowledge, from the existing literature. Their extensions for the arbitrary probability space and, further, to the continuous-time setting, require more sophisticated tools and will be published elsewhere.

The reader is invited to compare the suggested approach to that of the important article by Jouini and Kallal (1995), conceptually different not only at the level of modeling (continuous-time setting with bid and ask prices) but also in the formulation of the no-arbitrage criteria, see the end of our note. An attempt to find the arbitrage pricing theorem (for the binomial model) can be found in the preprint (Shirakawa and Konno, 1995).

Do not assuming that the reader has any background in random processes above the definitions and elementary properties of martingales and supermartingales, we explain standard (and very convenient) notations of stochastic calculus used throughout the paper. Namely, for $X = (X_t)$ and $Y = (Y_t)$ we define $X_- := (X_{t-1})$, $\Delta X_t := X_t - X_{t-1}$, and, at last,

$$X \cdot Y_t := \sum_{k=0}^{t} X_k \, \Delta Y_k,$$

for the discrete-time integral (here X and Y can be scalar or vector-valued). For finite Ω, if X is a predictable process (i.e. X_- is adapted) and Y is a martingale, then XY is a martingale. The product formula $\Delta(XY) = X\,\Delta Y + Y_-\,\Delta X$ is obvious. The books, Rockafellar (1970) and Pshenychnyi (1980), may serve as references in convex analysis.

2. Market with transaction costs

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space equipped with a discrete-time filtration $F = (\mathcal{F}_t)_{0\le t\le T}$ and an adapted d-dimensional process $S = (S_t)$, $S_0 = s$, having the strictly positive components and describing the dynamics of prices of d assets, e.g. currencies quoted in some reference asset (say, in "euro"). We assume that $\mathcal{F}_0$ is trivial.

We consider a model with proportional transaction costs given by a $d \times d$ matrix $\Lambda = (\lambda^{ij})$ with $\lambda^{ij} \ge 0$ and $\lambda^{ii} = 0$.

The agent's positions at time t can be described either by the vector of their values in "euros":

$$V_t = (V_t^1, \dots, V_t^d),$$