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The level crossing rates and associated statistical properties of a random frequency response function

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ABSTRACT

This work is concerned with the statistical properties of the frequency response function of the energy of a random system. Earlier studies have considered the statistical distribution of the function at a single frequency, or alternatively the statistics of a band-average of the function. In contrast the present analysis considers the statistical fluctuations over a frequency band, and results are obtained for the mean rate at which the function crosses a specified level (or equivalently, the average number of times the level is crossed within the band). Results are also obtained for the probability of crossing a specified level at least once, the mean rate of occurrence of peaks, and the mean trough-to-peak height. The analysis is based on the assumption that the natural frequencies and mode shapes of the system have statistical properties that are governed by the Gaussian Orthogonal Ensemble (GOE), and the validity of this assumption is demonstrated by comparison with numerical simulations for a random plate. The work has application to the assessment of the performance of dynamic systems that are sensitive to random imperfections.

1. Introduction

The prediction of the response of a linear system to harmonic forcing is straight forward in principle: the governing equations of most system components are well established, and highly sophisticated computer software is widely available to provide numerical solutions to these equations. A difficulty arises however if the system properties are random or otherwise uncertain, since this requires the uncertainty to be quantified and then propagated through the system equations of motion to yield the uncertainty in the response. Both aspects of this procedure are problematical — firstly it can be difficult or impossible to fully identify and quantify the system uncertainties (for example, in terms of probability density functions, or bounded variables), and secondly the propagation of the uncertainty can require very significant computational resources. For these reasons there is considerable interest in alternative methods of predicting the uncertainty in the harmonic response of an uncertain linear system, and in fact this has been a topic of research for more than half a century.

Early progress in predicting the response statistics of complex systems was made in the field of room acoustics by Schroeder and his co-workers. By 1954 expressions had been derived for a number of important statistical properties of the frequency response function (FRF) of the pressure in a room, including the mean value of the trough-to-peak height and the mean spacing between peaks [1,2]. It was assumed on the basis of the Central Limit Theorem (CLT) that the pressure has a complex-Gaussian distribution, and further work along these lines then considered the frequency-correlation function of the pressure FRF [3]. The application of the CLT rests on the assumption that many modes contribute to the response at any one

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frequency, and this requires a parameter known as the modal overlap to be high (typically greater than 3). The frequency beyond which this condition is met is known as the Schroeder frequency. In practice there is considerable interest in the statistical properties of an FRF below the Schroeder frequency, and later authors have considered this problem. In 1969 Lyon [4] developed a theory for the variance of the power input to a random structural or acoustic system, based initially on the premise that the system natural frequencies form a Poisson point process, but then with some allowance for effect of "repulsion" between natural frequencies. Lyon and his co-workers later considered the statistics of the phase of an FRF, and in particular the way in which the poles and zeros of the FRF affect the mean phase progression (for example [5,6]). Research in random matrix theory [7] throughout the 1970s and 1980s led to the realisation that the modal statistics of a highly random dynamic system might obey a universal distribution, and this fact was highlighted in 1989 by Weaver [8]. The universal distribution is known as the Gaussian Orthogonal Ensemble (GOE), with the name arising from a particular type of random matrix that is amenable to the derivation of closed-form eigenvalue statistics [7]. Much subsequent work on the topic of random FRFs has exploited the occurrence of GOE statistics [9], and expressions are available (for example) for the variance of both the frequency response [10,11] and the band-averaged response [12] of system components. The approach has also been combined with Statistical Energy Analysis (SEA) methods to yield the variance of the response of complex built-up systems [13,14]. As described below, the aim of the present work is to extend existing results to further properties of a random FRF. For completeness, it should be noted that the occurrence of GOE statistics requires the system to be sufficiently random, and if this is not the case then parametric uncertainties (for example [15]) may need to be considered, or alternatively the maximum entropy approach developed by Soize [16] might be applied. The present work is exclusively concerned with systems that might reasonably be assumed to display GOE statistics.

Previous work on the application of the GOE to FRF statistics has tended to focus on the prediction of the mean and variance, and ultimately the probability density function, of the response at a single frequency. This enables the probability that the response will exceed a particular level at a given frequency to be estimated, and confidence bands for the response can be established [17]. More general statistical questions that cannot be answered by using "single frequency" statistics include: (i) the probability that the FRF will exceed a particular level at least once within a given frequency band, and (ii) the average number of crossings of a particular level within a given frequency band. These questions would be of practical interest, for example, if the frequency of excitation is uncertain but known to lie within a particular band. These questions are addressed in the present work for the case of the FRF of the energy of a random system, and related questions such as the mean rate of the occurrence of peaks, and the mean trough-to-peak height are also addressed. The work makes extensive use of previous results from the random vibration literature concerning the mean rate at which a random process crosses a critical level [18]; this requires a new result to be derived for the variance of the slope (i.e. the frequency derivative) of the FRF and also for the *conditional* variance of the slope, meaning the variance at a prescribed value of the FRF. The prediction of the mean rate of the occurrence of peaks, and the mean trough-to-peak height, requires knowledge of the variance of the second derivative of the FRF, and again a new result is derived here for this quantity.

Section 2 of what follows presents background material regarding the FRF of the energy of a random system; the mean rate at which this function crosses a specified level is then derived in Section 3. Various other statistical properties of the function are then derived in Section 4, namely: the probability of crossing a specified level at least once, the mean rate of the occurrence of peaks, and the mean trough-to-peak height. The theoretical results are then compared in Section 5 with numerical results for a random plate.

2. Random frequency response functions

If a structural dynamic system is subjected harmonic forcing at frequency ω and has proportional damping (or if the damping can be reasonably approximated as being proportional) then the complex amplitude of a displacement response u at some location $\mathbf{x}$ within the system can be written in the well-known form [19].

$$u(\omega,\mathbf{x}) = \sum_n \frac{i\omega g_n \phi_n(\mathbf{x})}{\omega_n^2 - \omega^2 + i\eta\omega_n^2}, \quad g_n = \int_R P(\mathbf{x})\phi_n(\mathbf{x})\mathrm{d}\mathbf{x}. \tag{1,2}$$

Here ω_n is the nth natural frequency of the system, ϕ_n is the associated mode shape (scaled to unit generalised mass), and $P(\mathbf{x})$ describes the spatial distribution of the applied loading so that g_n is the generalised force acting in the nth mode. The time averaged kinetic energy of the system can be written as

$$T(\omega) = (1/4)\int_R \rho(\mathbf{x})|u(\omega,\mathbf{x})|^2 \mathrm{d}\mathbf{x}, \tag{3}$$

where $\rho(\mathbf{x})$ is the mass density and R is the spatial region occupied by the system. For ease of notation Eqs. (1)–(3) are explicitly concerned with the case of a scalar displacement variable and scalar mode shapes, although these results could readily be generalised to the case of vector quantities if required. It follows from Eqs. (1)–(3), and the fact that the mode shapes are mass-orthogonal, that the kinetic energy of the system can be written in the form