MÖBIUS-TODA HIERARCHY AND ITS INTEGRABILITY

Chuanzhong LI (李传忠)
Department of Mathematics, Ningbo University, Ningbo 315211, China
E-mail: lichuanzhong@nbu.edu.cn

Abstract In this paper, we construct a new integrable equation called Möbius-Toda equation which is a generalization of $q$-Toda equation. Meanwhile its soliton solutions are constructed to show its integrable property. Further the Lax pairs of the Möbius-Toda equation and a whole integrable Möbius-Toda hierarchy are also constructed. To show the integrability, the bi-Hamiltonian structure and tau symmetry of the Möbius-Toda hierarchy are given and this leads to the existence of the tau function.

Key words Möbius-Toda equation; soliton solutions; Lax equation; Möbius-Toda hierarchy; tau function

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1 Introduction

The Toda lattice equation is a completely integrable system which has many important applications in mathematics and physics including the theory of Lie algebra and random matrix model [1–5]. Toda system has many kinds of reduction or extension for example extended Toda hierarchy (ETH) [6], bigraded Toda hierarchy (BTH) [7–14] and so on. These Toda hierarchies have important application in Gromov-Witten theory on $\mathbb{C}P^1$ and orbiford.

The $q$-calculus traces back to the early 20th century and attracted important works in the area of $q$-calculus [15]. The $q$-deformation of classical nonlinear integrable system started in 1990’s by means of $q$-derivative $\partial_q$ instead of the usual derivative with respect the spatial variable in the classical system. Several $q$-deformed integrable systems have been presented, for example the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy is a subject of intensive studies in the literatures [16–22]. The $q$-Toda equation was also studied in [23, 24] but not for a whole hierarchy. In [13], we generalize the $q$-Toda equation to a whole hierarchy and give the bi-Hamiltonian structure and tau symmetry. Later we construct a extended $q$-Toda hierarchy and multicomponent $q$-Toda hierarchy in [14]. Basing on a generalization of the $q$-shift operator, this paper will be devoted to the further studies on a kind of Möbius-Toda hierarchy (MTH).

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To show the complete integrability of nonlinear evolution, we are going to test whether the Möbius-Toda equation has Hirota bilinear equation, three-soliton solution, Lax pair, bi-Hamiltonian structure and tau symmetry. This paper will show the integrability on the Möbius-Toda hierarchy from the above several directions.

2 Möbius-difference Operator

In the common sense, an integrable equation can always be rewritten in form of an Hirota bilinear equation using Hirota direct method. Therefore first we introduce some basic notation including Hirota derivatives as a preparation for introducing the Hirota bilinear equation of the Möbius-Toda equation.

Let $G$ be a space of differentiable functions $f, g : \mathbb{R}^n \to \mathbb{R}$. The Hirota $D$-operator $D : G \times G \to G$ is defined as

$$[D_x^m D_t^m \cdots] f \cdot g = [(\partial_x - \partial_{x'})^m (\partial_t - \partial_{t'})^m \cdots] f(x, t, \cdots) g(x', t', \cdots)|_{x' = x, t' = t, \cdots}$$

(2.1)

The virtue of a Hirota exponential identity can appropriately be as the following form in terms of the Hirota $D$-operator

$$e^{hD_x} f(x) g(x) = f(x + h)g(x - h).$$

(2.2)

If $h$ is a parameter and $f, g$ are continuous differentiable functions, like in [24], we define

$$\sigma_h(x) = e^{h(x)} \partial_x x,$$

(2.3)

then

$$e^{h\chi(x)} \partial_x u(x) = u(e^{h(x)} \partial_x x) = u(\sigma_h(x)), \ h > 0.$$  

(2.4)

If $\sigma_h(u(x)) = e^{h\chi(x)} \partial_x u(x) = u(x + h)$, the system will lead to the original Toda lattice. If $\sigma_h(x) = e^{h(x)} x = e^h x$, i.e., $e^{h(x)} \partial_x u(x) = u(e^h x)$ then the system will lead to the $q$-Toda lattice in [24].

Then we can define an Möbius operator $\Lambda_M$ with a non-vanishing parameter $c$ as following which will be used to define the Möbius-Toda lattice surveyed in this paper

$$\Lambda_M f(x) = e^{cD(x)} f(x) = f \left( \frac{\rho}{M(x) + c} + \xi \right) = f \left( \frac{ax + b}{cx + d} \right),$$

(2.5)

where

$$M(x) = \frac{\rho}{x - \xi}, \ \rho = \frac{a + d}{2}, \ \xi = \frac{2b}{d - a}, \ 4bc + (a - d)^2 = 0.$$  

(2.6)

In the above Möbius transformation we can always use the condition $ad - bc = 1$. Thus only two independent parameters $b$ and $c$ (the same number of parameters as in the definition of function $M(x)$). Moreover all parameters $a, b, c, d$ are assumed to be real-valued and we have $bc \leq 0$. As a consequence the expressions for the dependent parameters $a$ and $d$ can be written as follows $a = \sigma + \theta \sqrt{bc}$ and $d = \sigma - \theta \sqrt{-bc}$, where $\sigma^2 = 1 = \theta^2$. This clearly indicates that in the limiting case of $c$ tending to zero the operator $\Lambda_M$ tends to the usual shift operator characterized by the step $h = \sigma b$. Then he following proposition can be easily derived.

**Proposition 2.1** The Möbius-exponential identity acts on arbitrary continuous differentiable functions $f(x), g(x)$ as the rule

$$e^{cD_M} f(x) g(x) = [\Lambda_M f(x)] [\Lambda_M^{-1} g(x)], \ x \in \mathbb{R},$$

(2.7)
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