From microscopic taxation and redistribution models to macroscopic income distributions

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ABSTRACT

We present here a general framework, expressed by a system of nonlinear differential equations, suitable for the modeling of taxation and redistribution in a closed society. This framework allows one to describe the evolution of income distribution over the population and to explain the emergence of collective features based on knowledge of the individual interactions. By making different choices of the framework parameters, we construct different models, whose long-time behavior is then investigated. Asymptotic stationary distributions are found, which enjoy similar properties as those observed in empirical distributions. In particular, they exhibit power law tails of Pareto type and their Lorenz curves and Gini indices are consistent with some real world ones.

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1. Introduction

The interest of physicists and mathematicians towards complex systems arising in social and economical sciences has been constantly growing in the last years, as attested by the number of published papers. Among the subjects these papers deal with, one finds opinion formation dynamics (see for example Refs. [1–5]), relaxation processes to steady wealth and income distributions [6–17], mechanisms of financial markets and other out-of-equilibrium economic and financial phenomena [18–21]. These topics share the common fact of referring to systems (populations) composed of a large number of interacting elements (individuals). And this is why methods and tools from statistical mechanics and kinetic theory have been and are being adapted and employed to investigate them.

In a recent paper, [22], one of the present authors introduced a general framework, suitable for the construction of models of taxation and redistribution in a closed society. This framework originates from a discrete active particle kinetic approach [23] and is expressed by a system of nonlinear ordinary differential equations. The equations are as many as the classes, each one characterized by its "average income", in which the population can be divided. Each equation gives the variation in time of the fraction of individuals belonging to a certain class. The framework provides a description of the evolution of the wealth[1] distribution over the population and aims at explaining emerging collective features, based on the knowledge of the individual interactions. As a case study, a specific model was also formulated in Ref. [22]. To this end, the general mathematical framework was exploited and a particular choice for the values of some of its parameters was made. The well-posedness of the model as well as the existence of two conserved quantities, corresponding to the total population and the global wealth, was then established. Several simulations were carried out for the case in which the number n of income classes (and of differential equations) is equal to 5. Specific attention was devoted in Ref. [22] to the differences, detectable from the shape of the long-time income distributions, among systems with different taxation rates. The result

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[1] The words income and wealth are used in this paper to designate the same concept.

was that increasing the difference between the maximum and the minimum tax rate leads, at the asymptotic equilibrium, to the growth of the middle classes, to the detriment of the poorest and the richest classes. This is a reasonable feature which encourages, in spite of the naiveness and roughness of the model, a thorough investigation. A natural observation is that the study of the model in the particular case with 5 income classes does not allow for example to recover the Pareto-law which is observed in real world economies. As a matter of fact, $n = 5$ is too small a number for a tail of the steady income distribution to show up. In view of that, we started performing a great deal of computational simulations relative to the model with greater values of n. We tried and considered several choices of the model parameters, expressing e.g. different characterizations of the incomes or of the tax rates. With the aim of treating reasonable cases, as far as possible comparable with real world ones, we focused our attention on "realistic" initial population distributions, where the majority of individuals belong to lower income classes, while higher classes are less densely populated (see Section 3 for details). The numerical solutions systematically show that for any fixed value of the total wealth a unique asymptotic stationary distribution exists, independently of the random choice of the initial population distribution (subjected only to the just mentioned "realistic" requirement). The asymptotic stationary distribution exhibits the following patterns: the density of the low income classes is smaller than for the low–medium classes, the maximal density is achieved by the low–medium classes and the density progressively decreases for the higher income classes. In fact, the asymptotic distributions exhibit salient features of empirical distributions (see e.g. Refs. [24, page 14], [25, page 19], and [26, page 8]). At a closer look, one also finds that the tails of the distributions have indeed a power law behavior [27]. An analysis of the shape of the tails and its relation with the parameters of the model is the subject of the present paper.

The paper is organized as follows. In Section 2, we review for the convenience of the reader the framework and the model introduced in Ref. [22]. Section 3 focuses on the existence of asymptotic stationary distributions and on their properties. In particular, Pareto tails [28] are found to occur and their Pareto indices are calculated in some different cases. The last section provides a short summary of the content of the paper and a brief mention of possible future developments.

Before starting, it may be of interest to compare certain features of our models with those of others available in the econophysics or "classical" mathematical economics literature. Some immediately evident differences concern the mathematical formulation of the income distribution problem. A first class of works (see Ref. [17] for a review) develop a statistical analysis of the population by means of Monte Carlo simulations of the interactions of a large number of individuals. In these models the interaction rules can be defined with great freedom, because they are applied in a straightforward way through the simulation algorithm. At the same time, the method is not based upon evolution equations. Therefore it lacks general mathematical theorems which should take into account the specific hypotheses on the interactions. To draw a comparison, within our framework we are free to fix several parameters (interaction frequency, taxation rates, etc.), but for the models to be conservative, some terms in the transition probabilities expressing income class changes must be proportional to the reciprocal of the income difference $|r_i - r_j|$. If one changes this dependence, supposing for instance that the transition probability is proportional to the reciprocal of $|r_i - r_j|^2$, then the conservation of the total wealth ceases to be valid. The dependence of the model on the details of the interactions is also typical of the approaches based on the Boltzmann equation [14,15]. In the application of this equation to the kinetic theory of gases the interactions are determined to a large extent by physical conservation laws and symmetry principles. In the applications to econophysics there is some more freedom as for the definition of the interactions. Within this approach it is possible to prove powerful general results concerning the moments of the distribution function $f(w, t)$ and the Pareto index of the stationary asymptotic distribution. It has to be noticed however that the structure of the Boltzmann equation requires advanced mathematical tools for his treatment: the partial time derivative of the income distribution function $f(w, t)$ is given by an integral operator acting on $f(w, t)$ and also involves an average on some stochastic variables. The time evolution is usually computed through some approximate discretization method. Similar approximations are also employed in the classical economics theory [29]. There, the time evolution equations are not derived from hypotheses on the "microscopic" interactions, but from variational principles and other general "macroscopic" considerations. Of course, the classical approach is characterized by a more realistic description of the dynamics of a complex economy, taking into consideration also different kinds of assets, financial transactions, government intervention etc.

The possible appearance of power law tails for heterogeneous kinetic wealth-exchange models was shown in Ref. [9] (see also Ref. [8]) to be deducible through a variational approach, based on the the minimization of the Boltzmann entropy functional. Heterogeneity refers here to a parameter expressing the saving propensity of the agents. The question of taxation and redistribution was studied with different approaches in Refs. [13,16]. In Ref. [16] two-step tradings are considered, comparable with inelastic binary "collisions", followed by the redistribution of a kind of lost energy, which represents the taxes. In Ref. [13] the authors refer to the Boltzmann equation. Both papers point out that the effect of subsidies by the government on the equilibrium distribution is that of shifting the individuals from the lower income classes toward middle income classes. Accordingly, and differently with respect to what happens e.g. to Boltzmann–Gibbs distributions, the equilibrium distribution exhibits a maximum. Figure 5 of Ref. [13] is qualitatively similar to the figures of this paper with the histograms representing the asymptotic equilibria. Also, in this connection, a sharp and narrow peak at low incomes observed in Ref. [6] in the plots of statistical data on personal income distribution in Australia was interpreted by the authors of Ref. [6] as the result of government policy about redistribution. To further stress some analogy with Ref. [13], we recall that the equation system (7) in the present paper essentially is a particular discrete version of the Boltzmann equation. Our framework introduces from the beginning a discretization of the distribution function. We suppose that the individuals belong to income classes and pass from one class to another with certain probabilities. In our case, the equivalent of the