Essential equilibrium in normal-form games with perturbed actions and payoffs

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1. Introduction

A Nash equilibrium of a normal-form game \(G\) is essential if it is robust to perturbations of \(G\). For generic games in the collection of all finite-action games with fixed action spaces, all Nash equilibria are essential (cf. Wu and Jiang, 1962). This result has been extended to infinite-action games (e.g., Yu, 1999, Carbonell-Nicolau, 2010, 2015, and Scalzo, 2013). Yu (1999) allows for perturbed action spaces and payoff functions, but requires continuity of payoff functions. Carbonell-Nicolau (2010, 2015) and Scalzo (2013) allow for discontinuous payoffs but require fixed action spaces. In this paper we extend the results in Carbonell-Nicolau (2010) by allowing for perturbed payoffs and actions.

The notion of perturbed game used in this note differs from the definition adopted in Yu (1999). We argue in Section 2 that, in the presence of payoff discontinuities, perturbing actions and payoffs as in Yu (1999) poses problems. In fact, under Yu’s approach it is easy to construct games whose perturbations do not include strategies that are of particular strategic significance to the players. Our discussion in Section 2 is framed in terms of a very simple example, which showcases the difficulties of the approach and illustrates the intuitive appeal of the definition of a perturbed game proposed here.

2. Preliminaries

A **normal-form game** (or simply a **game**) \(G = (X_i, u_i)_{i \in N}\) consists of a finite number \(N\) of players, a nonempty set of actions \(X_i\) for each player \(i\), and a payoff function \(u_i : X \rightarrow \mathbb{R}\) for each player \(i\) defined on the set of action profiles \(X := \times_{i \in N} X_i\).

For each player \(i\), let \(X_i\) be a nonempty, compact, convex subset of a metric vector space. Let \(X := \times_{i \in N} X_i\) be endowed with the associated product topology. The sets \(X_1, \ldots, X_N\) will be fixed throughout the analysis. Let \(B(X)\) denote the set of bounded maps \(f : X \rightarrow \mathbb{R}\). Let \(K(X_i)\) denote the hyperspace of nonempty, compact, and convex subsets of \(X_i\). Define

\[
G_X := \left( \times_{i \in N} K(X_i) \right) \times B(X)^N.
\]

A typical member of \(G_X\) is denoted \((Y, u) = (Y_1, \ldots, Y_N, u_1, \ldots, u_N)\) and can be viewed as a normal-form game \((Y, u) := \{ (y_i, u_i) \}_{i \in N} \times \times_{i \in N} Y_i\).

In Yu (1999), the space \(B(X)^N\) is endowed with the metric \(\gamma_X : B(X)^N \times B(X)^N \rightarrow \mathbb{R}\) defined by

\[
\gamma_X ((u_1, \ldots, u_N), (v_1, \ldots, v_N)) := \sum_{i \in N} \sup_{x \in X} |u_i(x) - v_i(x)|,
\]

and, for each \(i\), the space \(K(X_i)\) is endowed with the Hausdorff metric topology. The associated product metric space \(G_X\), endowed with the corresponding product topology, constitutes the space of games considered in Yu (1999). This topology defines the notion of perturbed game used in Yu (1999), and we wish to argue here that this notion is not appropriate in the presence of payoff discontinuities. To illustrate, consider the one-person game \(([0, 1), u)\), where \(u(x) := 0\) if \(x \in [0, 1)\) and \(u(1) := 1\), and the sequence...
(0, 1 − 1/n, u), which converges to (0, 1, u). Arguably, the strategy x = 1, which dominates every other strategy, is particularly relevant in this game, and it seems hard to justify an approximation that does not include this strategy or another strategy that plays a similar role. In particular, the games (0, 1, u) and (0, 1 − 1/n, u) appear markedly dissimilar, even for large n, and the sequence (0, 1 − 1/n, u) does not seem to well-approximate (0, 1, u).1 By contrast, the sequence (0, 1 − 1/n, v^n), where v^n(x) := 0 if x ∈ [0, 1 − 1/n] and v^n(x) := 1 − 1/n if x ∈ [1 − 1/n, 1], seems to better approximate (0, 1, u) (for large n). Note that for the above topology, while the sequence (0, 1 − 1/n, u) converges to (0, 1, u), the sequence (0, 1 − 1/n, v^n) does not converge to (0, 1, u). In the next paragraph, we define a topology that is consistent with the idea that games of the form (0, 1 − 1/n, u) are close to (0, 1, u) (for large n) while games of the form (0, 1 − 1/n, v^n) are not.2

Given i and {Y_i, Z_i} \subseteq K(X_i), let H(Y_i, Z_i) be the set of all homeomorphisms h_i from Y_i to Z_i such that h_i(A) \subseteq Z_i is convex if and only if A \subseteq Y_i is convex. Let d_{X_i} be a compatible metric for X_i. Let \Theta_X represent the set of normal-form games (Y_i, u_i)_{i \in N} such that (Y_i, u_i) ∈ G_X. Note that a member of G_X uniquely determines a corresponding element of \Theta_X, while there is a one-to-many mapping between \Theta_X and G_X. For the members of \Theta_X, we write (Y_i, u_i)_{i \in N} = (Y, u) \in \Theta_X and (Y_i, u_i)_{i \in N} = (Y, u) \in \Theta_X indisputably, which entails a slight abuse of notation. Define the map \alpha_X : \Theta_X × \Theta_X \rightarrow R ∪ {∞} by

\alpha_X((Y, u), (Z, v)) := \inf \left\{ \epsilon > 0 : \exists h \in \times_{i \in N} H(Y_i, Z_i) : \right.

\left. \sup_{x \in Y} \left| u_i(x) - v_i(h(x)) \right| \leq \epsilon \text{ and } \sup_{x \in Y} d_{X_i}(h(x), x) \leq \epsilon \right\}

if \times_{i \in N} H(Y_i, Z_i) \neq \emptyset, and \alpha_X((Y, u), (Z, v)) := \infty if \times_{i \in N} H(Y_i, Z_i) = \emptyset. Now define the metric \rho_X : \Theta_X × \Theta_X \rightarrow R by \rho_X((Y, u), (Z, v)) := \min \{\alpha_X((Y, u), (Z, v)), 1\}. Throughout the sequel, we endow \Theta_X with the metric \rho_X.

Remark 1. As illustrated by the previous example, the metric \rho_X differs from the Yu metric. This discrepancy can be even be found within the subdomain of continuous games. Indeed, for X := [0, 1] and arbitrary u, the sequence of games (0, 1 − 1/n, u) in G_X converges to (0, 1, u) in the sense of Yu, and yet this sequence does not converge with respect to \rho_X because none of its members is homeomorphic to the game (0, 1, u). Thus, convergence in the sense of Yu need not imply convergence with respect to \rho_X.

Definition 1. A correspondence \Phi : A \rightrightarrows B between topological spaces is \textit{upper hemicontinuous} at x \in A if the following condition is satisfied: for every open set V \subseteq B with V ∩ Φ(x) ≠ ∅ there is a neighborhood V_x of x such that y ∈ V_x implies \Phi(y) ∩ V ≠ ∅. \Phi is \textit{lower hemicontinuous} if it is lower hemicontinuous at every point in A.

Definition 2. A correspondence \Phi : A \rightrightarrows B between topological spaces is \textit{lower hemicontinuous at} x \in A if the following condition is satisfied: for every open set V \subseteq B with V ∩ Φ(x) ≠ ∅ there is a neighborhood V_x of x such that y ∈ V_x implies \Phi(y) ∩ V ≠ ∅. \Phi is \textit{lower hemicontinuous} if it is lower hemicontinuous at every point in A.

Definition 3. A strategy profile x = (x_i, x_{i−1}) in X is a \textbf{Nash equilibrium} of G = (X, u) relative to \Theta_X if for every neighborhood V_x of x there is a neighborhood V_{x_i} of x_i such that (x_i, x_{i−1}) ∈ V_x and all x_j, j ≠ i are in V_j. Given a family of games \Theta \subseteq \Theta_X, the restriction of \xi_X to \Theta is denoted by \xi_X|_{\Theta}.

Definition 4. Given a class of games \Theta \subseteq \Theta_X, a Nash equilibrium x of (Y, u) ∈ \Theta is an \textit{essential equilibrium} of (Y, u) relative to \Theta if for every neighborhood V_y of y there is a neighborhood V_{y−1} of y−1 such that for every (z, f) ∈ V_y \times V_{y−1} \in \Theta, V_y \times \xi_{\Theta}(z, f) ≠ ∅.

Definition 5. Suppose that \Theta \subseteq \Theta_X. A game (Y, u) ∈ \Theta is \textit{essential relative} to \Theta if every pure-strategy Nash equilibrium of (Y, u), (Y, u) is essential relative to \Theta. When the domain of reference is clear from the context, we shall simply say that (Y, u) is an \textit{essential game}.

Remark 2. Suppose that \Theta \subseteq \Theta_X. A game (Y, u) ∈ \Theta is essential relative to \Theta if and only if \xi_X|_{\Theta} is lower hemicontinuous at (Y, u).

3. The results

The following definition was introduced in Barelli and Soza (2009).

\[ h^2 = \times_{i=1}^N h(Y_i', Y_i'') \]

\[ \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(h^2(h_i(x)))) = \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(x)) \]

\[ \leq \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(x)) \]

\[ + \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(x)) \]

\[ = \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(x)) \]

\[ + \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(h_i(x))) \]

\[ \leq \sum_{i=1}^N \sup_{x \in Y_i} (u_i(x) - u_i'(h_i(x))) \]

where \( u_i' : Y_i' \rightarrow Y_i' \) and \( u_i'' : Y_i'' \rightarrow Y_i'' \) are defined by

\[ u_i'((x_i)) := u_i'(h_i^2(u_i(x_i))) \] and \[ u_i''((x_i)) := u_i''(h_i(h^2(u_i(x_i)))) \]

and

\[ \sup_{x \in Y} d_h(h_i^2(u_i(x)), x) \leq \sup_{x \in Y} d_h(h_i(x), x) + \sup_{x \in Y} d_h(h^2(h_i(x)), h_i(x)) \]

Consequently, \[ \alpha_X((Y, u), (Y', u')) \leq \alpha_X((Y, u'), (Y', u')) + \alpha_X((Y', u'), (Y', u'')) \] and so \[ \rho_X((Y, u), (Y', u')) \leq \rho_X((Y, u'), (Y', u')) + \rho_X((Y', u'), (Y', u'')) \]. Thus, \( \rho_X \) is indeed a metric on \( \Theta_X \).

\footnote{The idea that “good” approximations to an infinite discontinuous game should include strategies that are of particular strategic significance to the players is already discussed in Simon (1987) and Reny (2011) in the context of finite strategic approximations to infinite games.}

\footnote{This is in fact an example in which a game with a dominant strategy can only be approximated, in the new topology, by games with a dominant strategy. This is obviously false about the Yu topology. We conjecture that this property holds in general, and we thank an anonymous referee for bringing up this point.

\footnote{It is easily seen that \( \rho_X((Y, u), (Z, v)) = 0 \) ⇔ (Y, u) = (Z, v) for all (Y, u) and (Z, v) in \( \Theta_X \). Also, it is clearly the case that \( \rho_X((Y, (Z, v)) = \rho_X((Z, (Y, v)) \) for all (Y, u) and (Z, v) in \( \Theta_X \). To verify that the triangle inequality holds for \( \rho_X \), fix (Y, u) \in \( \Theta_X \), and (Y, v), (Z, W) in \( \Theta_X \) and note that given h^i ∈ \times_{i=1}^N h(Y_i, V_i) and h^j ∈ \times_{j=1}^N h(Y_j, V_j), where (Y, u) := (Y', u') \in \Theta_X \).}}
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