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A practical approach to sensitivity analysis in linear programming under degeneracy for management decision making

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ABSTRACT

Linear programming (LP) is a widely used tool in management decision making. Theoretically, sensitivity analysis of LP problems provides useful information for the decision maker. In practice, however, most LP software provides misleading sensitivity information if the optimal solution is degenerate. The paper shows how sensitivity analysis of LP problems can be done correctly when the optimal solution is degenerate. A production planning example is presented to illustrate the incorrect sensitivity analysis results automatically provided by most LP solvers. The general characteristics of the misleading results and the possible effects of this incorrect information on management decisions are also discussed.

1. Introduction

One of the most important management decision making problems is, when limited amount of resources must be assigned to decision alternatives and an objective function helps to evaluate the result of the assignment. Profit maximizing product mix decisions, cost minimizing production planning problems are typical examples of this situation. If the objective function is linear and the limits on resource usage are expressed with linear inequalities, furthermore, the decision variable is continuous, a continuous linear programming problem (LP) is obtained.

The standard form of a primal linear programming problem is the following (Hillier and Lieberman, 1995):

$$\mathrm{Max}(\underline{\mathbf{c}}^{\mathrm{T}}\underline{\mathbf{x}}) \qquad \mathbf{A}\underline{\mathbf{x}} \le \underline{\mathbf{b}} \qquad \underline{\mathbf{x}} \ge 0. \tag{1}$$

The standard form of the dual linear programming problem belonging to (1) is as follows:

$$\mathrm{Min}(\underline{\mathbf{b}}^{\mathrm{T}}\underline{\mathbf{y}}) \qquad \mathbf{A}^{\mathrm{T}}\underline{\mathbf{y}} \ge \underline{\mathbf{c}} \qquad \underline{\mathbf{y}} \ge 0. \tag{2}$$

Notations used in the paper are summarized in Table 1.

The optimal solution of an LP problem consists of two parts. The optimal solution of the primal problem ($\underline{\mathbf{x}}^*$) provides information about the optimal allocation of limited resources. The dual optimum ($\underline{\mathbf{y}}^*$) provides information about the marginal change of the objective function if a right-hand-side parameter changes.

Sensitivity analysis of the optimal solutions can provide further useful information for management. Sensitivity information consists of the validity ranges of the primal and of the dual optimum. Two types of ranges are calculated. Validity ranges of the *objective function coefficients* (OFC) provide a range for each coefficient. Within this range the *primal* optimal solution will not change. Validity ranges of the *right-hand-side (RHS) elements* provide a range for each right-hand-side element. Within this range the *dual* optimum will not change.

If the optimal solution degenerates the implementation of the optimal solution and the interpretation of sensitivity analysis results raise several problems (Rubin and Wagner, 1990). In case of degeneracy the information necessary for management decision making, and the information about the mathematical properties of the optimal solution are different; therefore, further calculations are necessary to get the appropriate information for management (Jansen et al., 1997).

The theoretical problems of sensitivity analysis under degeneracy are well known in the literature. Several papers discuss the reason on degeneracy and present methods for calculating special sensitivity information (see, for example, Evans and Baker, 1982; Gal, 1986; Jansen et al., 1997). Many papers are published to demonstrate erroneous management decisions based on the misinterpreted results of sensitivity analysis. Rubin and Wagner (1990) use a transportation problem to show how shadow prices can be misleading. Jansen et al. (1997) use a refinery problem to demonstrate that almost half of the ranging information is misleading. A production planning example is used by Koltai and Terlaky (2000) to illustrate the obvious errors of sensitivity results, when this information is used for management decision making.

Two types of problems must be differentiated. *First*, the interpretation of shadow price is not clear when the problem is primal degenerate. Aucamp and Steinberg (1982) explain the

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Table 1
Summary of notation.

$\mathbf{A}$	Coefficient matrix with elements a_{ji} (j=1,…, J; i=1,…,I)
$\underline{\mathbf{b}}$	Right-hand-side vector with elements b_j (j=1,…,J)
$\underline{\mathbf{c}}$	Objective function coefficient vector with elements c_i (i=1,…,I)
$\underline{\mathbf{x}}$	Variable vector of the primal problem with elements x_i (i=1,…,I)
$\underline{\mathbf{x}}^*$	Optimal solution of the primal problem with elements x_i^* (i=1,…,I)
$\underline{\mathbf{y}}$	Variable vector of the dual problem with elements y_j (j=1,…,J)
$\underline{\mathbf{y}}^*$	Optimal solution of the dual problem with elements y_j^* (j=1,…,J)
OF^*	Optimal value of the objective function
$\underline{\mathbf{e}}_i$	Unit vector with I elements, and with e_i=1 and e_k=0 for all $k \neq i$
$\underline{\mathbf{e}}_j$	Unit vector with J elements, and with e_j=1 and e_k=0 for all $k \neq j$
δ	Perturbation of a right-hand-side parameter
y_j^-	The left shadow price of right-hand-side element b_j ($\delta < 0$)
y_j^+	The right shadow price of right-hand-side element b_j ($\delta > 0$)
γ_i	Change of objective function coefficient c_i
γ_i^-	Feasible decrease of objective function coefficient c_i
γ_i^+	Feasible increase of objective function coefficient c_i
ξ_j	Change of right-hand-side element b_j
$n\xi_j^-$	Feasible decrease of b_j belonging to the left shadow price
$n\xi_j^+$	Feasible increase of b_j belonging to the left shadow price
$p\xi_j^-$	Feasible decrease of b_j belonging to the right shadow price
$p\xi_j^+$	Feasible increase of b_j belonging to the right shadow price
n_t	Number of working days in month t
K	Productivity constant
H_t	Number of workers hired in month t
h_t	Unit cost of hiring in month t
F_t	Number of workers fired in month t
f_t	Unit cost of firing in month t
I_t	Inventory level in month t
i_t	Unit cost of inventory holding in month t
W_t	Number of workers in month t
D_t	Demand in month t

different interpretation of shadow prices and show, how different values can be distracted from the simplex table when the optimal solution is degenerate. Akgül (1984) suggests that the increase and the decrease of an RHS parameter must be differentiated. He provides simple examples to demonstrate the significance of the application of left and right shadow prices. *Second*, the sensitivity range of OFCs in case of primal degeneracy, and the validity range of the RHS elements under dual degeneracy are ambiguous. The mathematical properties of these ranges are thoroughly discussed by Gal (1986), and a graphical interpretation is given by Evans and Baker (1982).

Koltai and Terlaky (2000) show that sensitivity analysis results provided by most commercial LP solvers are not erroneous, just the objective of analysis must be correctly defined. Traditional sensitivity analysis is mathematical oriented and provides information about the validity of an optimal basis (Type I sensitivity). Two other types of sensitivity ranges can be defined for managers. Type II sensitivity provides information about the optimality of a support set, and Type III sensitivity provides information about the invariance of the rate of change of the objective function. These definitions are further refined for interior point algorithms by Hadigheh and Terlaky (2006). The calculation of Type II sensitivity, and under special conditions for Type III sensitivity as well are presented by Hadigheh and Terlaky (2006) and Hadigheh et al. (2007).

Since the solution of LP problems is increasingly an everyday tool of managers (Caine and Parker, 1996), and excellent and easy to use tools exist for solving LP problems (Raggsdale, 2007), the correct calculation of the sensitivity results is very important.

It can be concluded that two different approaches exist in the literature. There are theoretical papers about the properties and the effects of degeneracy, and there are practice oriented papers discussing the misleading effect of degeneracy in management decision making. There is no paper, however, which helps the decision maker to get the correct results for all relevant parameters. The novelty of this paper is to provide a bridge between the existing practical problems and the accumulated theoretical results. A unified framework is provided to set up all the additional LP problems necessary to solve for management oriented sensitivity results. The implementation of the calculation, the interpretation of the results and the application of the provided information in management decision making are also discussed.

The paper is structured as follows. First, a calculation method is suggested to get proper sensitivity analysis results for management decision making when the optimal solution is degenerate. Next, a small problem is provided to illustrate the misleading sensitivity information provided by commercially available LP solvers. A production planning case study is also presented to illustrate the problems of management decision making if sensitivity results are not correct. Reduction possibilities of computational burden and filtering of sensitivity results to enhance important information are also discussed. Finally, the conclusion summarizes the types of mistakes managers may commit based on erroneous sensitivity information and highlights the potential benefits of the suggested calculation.

2. Sensitivity analysis under degeneracy

Under degeneracy three types of problem must be faced:

1. When an optimal solution is primal degenerate several optimal bases provide the same optimal values for the decision variables. Management oriented OFC sensitivity information provides a range, within which the optimal *values* of the decision variables are unchanged. For example, the same quantities are optimal in a product mix. The rate of change of the optimal value function with respect to OFC c_i is equivalent to the optimal value of decision variable x_i^*. As long as this optimal value is unchanged, the rate of change of the optimal value function is the same. Therefore, for management decisions, the correct sensitivity range of an OFC is equivalent to the linearity interval of the optimal value function.
2. When the optimal solution is dual degenerate several primal optimal solutions may belong to the same dual optimum. Management oriented RHS sensitivity information provides a range, within which the optimal value of the dual variables is unchanged. For example, the same shadow price is optimal for a bottleneck resource, independently of the quantities produced. The rate of change of the optimal value function with respect to RHS b_j is equivalent to the optimal value of decision variable y_j^*. As long as this optimal value is unchanged, the rate of change of the objective value function is the same. Therefore, for management decisions, the correct sensitivity range of an RHS parameter is equivalent to the linearity interval of the optimal value function.
3. Since primal degeneracy is the result of more than necessary binding constraints in the optimum solution, for some constraints the effect of increase and the effect of decrease on the objective function value are different. In this case, the shadow price given by the LP solvers provides either half of the information (the effect of increase or the effect of decrease) or totally irrelevant information. The shadow price information is irrelevant from management point of view if the value of an RHS parameter is at a break point of the piecewise linear optimal value function, and the shadow price is a sub-differential with 0 validity range.

To get those shadow prices and validity range which are useful for management decision making additional LP problems must be solved. These additional LP problems are summarized in Table 2.