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Discrete versus continuous time models: Local martingales and singular processes in asset pricing theory

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ABSTRACT

In economic theory, both discrete and continuous time models are commonly believed to be equivalent in the sense that one can always be used to approximate the other, or equivalently, any phenomena present in one is also present in the other. This common belief is misguided. Both (strict) local martingales and singular processes exist in continuous time, but not in discrete time models. More importantly, their existence reflects real economic phenomena related to arbitrage opportunities, large traders, asset price bubbles, and market efficiency. And as an approximation to trading opportunities in real markets, continuous trading provides a better fit and should be the preferred modeling approach for asset pricing theory.

1. Introduction

It is commonly believed that in economic theory both discrete and continuous time models are equivalent in the sense that one can always be used to approximate the other, or equivalently, any economic phenomena present in one is also present in the other. Unfortunately, this common belief is misguided. Both (strict) local martingales and increasing singular processes exist in continuous time models, but not in discrete time models. These processes arise naturally in the mathematics of

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stochastic integration theory. For example, Itô integrals with respect to Brownian motion may not be martingales, but they are always local martingales. Such integrals arise when calculating the results of a hedging strategy, integrated against a price process.

Local times for Brownian motion are examples of naturally occurring increasing processes that have paths that are singular with respect to Lebesgue measure. An example of how local times can arise is when certain kinds of convex functions (including the function $f(x) = (x - K)_+$) are composed with a price process, as happens when describing certain kinds of options. The reason this is important is that the absence of arbitrage is essentially equivalent to the existence of a risk neutral measure that makes the price process a local martingale (for processes with continuous paths), and in the Brownian paradigm the presence of a local times makes this procedure impossible.[2]

This would not be a concern except for the fact that these processes characterize real and important economic phenomena related to arbitrage opportunities, large traders, asset price bubbles, and market efficiency. Discrete time models will not exhibit these phenomena. In addition, as an approximation to current security markets, continuous trading provides a better fit. This is true because a trade can take place at any time during a day, and not on a fixed and predetermined grid. Consequently, at least for asset pricing theory, continuous time is the preferred modeling approach. The purpose of this paper is to clarify and to justify these previous assertions.

An outline for this paper is as follows. Section 2 quickly explains how local martingales and singular processes are generated in continuous time models, and it characterizes the economic phenomena they represent. Section 3 discusses why continuous time models provide the better approximation to actual security markets, while Section 4 concludes.

2. Local martingales and singular processes

In a continuous time model, local times are examples of increasing processes which are singular with respect to Lebesgue measure (see Protter, 2005), however singular processes can arise through other means, such as in the study of drawdowns (see for example the recent work of Hadjiliadis and Zhang (in press)). Such processes do not exist in discrete time models. In stochastic integration theory, Itô integrals with respect to Brownian motion are local martingales, but they are not always martingales. The existence of local martingales is crucial to the general theory of stochastic integration. In discrete time, P.A. Meyer showed in 1973 that local martingales must be martingales. Kabanov has given an elegant treatment of this and a related idea in Kabanov (2008). Hence, strict local martingales do not exist in discrete time, except in a certain generalized sense. This is only a concern, however, if local martingales represent real and important economic phenomena. This is indeed the case, as we will next explain.

2.1. No free lunch with vanishing risk (NFLVR)

Martingales were first emphasized in asset pricing theory via the seminal papers of Harrison and Pliska (1981) and Harrison and Kreps (1979). For a bounded stochastic asset price process in discrete time, they showed that the absence of arbitrage, naturally defined, was equivalent to the existence of an equivalent[3] probability measure such that a normalized asset price under this new measure is a martingale. These results were extended to continuous time, within the framework of stochastic integration and Brownian motion filtrations by Harrison and Pliska (1983). Subsequently, a long quest was begun to find necessary and sufficient conditions for arbitrage-free price processes for more general stochastic processes. Building on the seminal work of Kreps (1981), success was finally achieved by Delbaen and Schachermayer (1994), Delbaen and Schachermayer (1998). Delbaen and Schachermayer formalized the absence of arbitrage with the notion of "no free lunch with vanishing risk (NFLVR)," and this was shown to be equivalent to the existence of an equivalent probability measure such that the normalized

[2] This is a consequence of Girsanov's theorem, that allows one to remove the drift of the price process through a change of measure procedure. One can show that a necessary condition for the removal of the drift (in the Brownian paradigm) is that the drift have paths that are absolutely continuous with respect to Lebesgue measure.

[3] Two probability measures P and Q are *equivalent* if they have the same sets of probability zero.