Greater parametric downside risk aversion

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A R T I C L E   I N F O

Article history:
Received 24 December 2016
Received in revised form 12 April 2017
Accepted 10 May 2017
Available online 25 May 2017

Keywords:
Downside risk aversion
Schwarzian
Protective puts
Self-protection

A B S T R A C T

We show that, just as an expected utility maximizer with utility function u responds to a compensated increase in risk by adjusting a control variable to reduce the degree of risk aversion measured by the Arrow–Pratt index $R_u = -u''/u'$ (Diamond & Stiglitz, 1974), so the response to a compensated increase in downside risk entails adjusting the control to reduce the degree of downside risk aversion measured by the Schwarzian $S_u = u''/u' - (3/2)u^2$. We also show that, ceteris paribus, increases in $S_u$ and in $R_u$ result in reduced exposure to downside risk and, therefore, greater demand for self-protection activities that reduce downside risk to future income. An increase from $S_u$ to $S_v$ is characterized by downside risk-averse transformations of utility everywhere along a path from $u$ to $v$, which together constitute what we define to be a parametric increase in downside risk aversion. These parametric increases yield comparative statics predictions not true if $v$ is simply a downside risk-averse transformation of $u$, and predictions for incremental changes in risk preferences can be extended immediately to global changes.

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1. Introduction

In relating increases in risk and in risk aversion, Diamond and Stiglitz (1974, Theorem 2) establish the fundamental result that the optimal choice for a control variable changes so as to reduce the degree of risk aversion in response to a compensated increase in risk, that is, a change in the distribution for income $y$ that induces a mean preserving spread in the distribution for utility $u(y)$. We extend this result from the second to the third order, showing that the optimal control changes to reduce the degree of downside risk aversion in response to a compensated increase in downside risk, one that preserves the mean and variance of utility.2

Diamond and Stiglitz also analyze the case in which the roles of the control and shift variables are reversed, interpreting the control as a preference parameter with the distribution chosen optimally from a family of risks. However, when the family is ordered by degree of risk, all risk averters choose the least risky distribution. By contrast, when ordered by degree of downside risk, an interior distribution may be optimal for some decision makers. We show that, in these instances, an increase in downside risk aversion and in risk aversion leads to a reduction in exposure to downside risk. In the context of self-protection activities that reduce exposure to future downside risk, the implication is that demand for these activities increases with increases in downside risk aversion and in risk aversion.

These conclusions are predicated on the degree of risk aversion for utility $u(y)$ being measured by the index of absolute risk aversion $R_u = -u''/u'$ introduced by Arrow (1970) and Pratt (1964), and on the degree of downside risk aversion being measured by the Schwarzian index $S_u = u''/u' - (3/2)u^2$, which we introduced in Keenan and Snow (2002). These two measures share an important mathematical property responsible for their centrality in the comparative statics predictions summarized above, namely, both satisfy 1-cocycle conditions that convert a composition mapping $\psi$, such as $v = \psi(u)$, into the addition operation through

\[ R_v - R_u = u'R_{\psi} \]  

(1)

in the case of risk aversion, and through

\[ S_v - S_u = [u']^2S_{\psi} \]  

(2)

in the case of downside risk aversion. These conditions ensure that rankings based on $R_u$ and on $S_u$ are transitive and are therefore capable of producing meaningful comparative statics predictions.3

2 After replacing $u$ with $u_{\psi_1, \psi}$ with $u_{\psi_1}$ and $v$ with $u_{\psi_2} = u_{\psi_1} \circ \psi_2$, conditions (1) and (2) show that rankings by $R_{\psi}$ and by $S_{\psi}$ are preserved under composition of

http://dx.doi.org/10.1016/j.jmateco.2017.05.002
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As shown in Theorem 1 of Keenan and Snow (2002), an infinitesimal change in risk preferences uniformly increases the Schwarzian if and only if the transformation of utilities is itself downside risk averse, just as the index of risk aversion increases uniformly if and only if the transformation is risk averse. Indeed, it seems natural to associate greater downside risk aversion with downside risk-averse transformations of utility as an intuitive extension of the association of greater risk aversion with risk-averse transformations. However, the equivalence of increases in the Schwarzian and downside risk-averse transformations (ones with ϕ'''' > 0) does not extend to large changes in risk preferences. In the large, such a transformation of utility does not continue to imply an increased Schwarzian measure, since unlike the ordering induced by the Schwarzian, the ranking of utility functions induced by downside risk-averse transformations is not even transitive, and so cannot serve as the basis for a coherent definition of greater downside risk aversion.

In this paper, we equate greater downside risk aversion with an increasing Schwarzian by requiring that utility ψ(y) differ from u(y) by a family of transformations each of which is downside risk averse, even at the infinitesimal level. We refer to this as a parametric increase in downside risk aversion. It is straightforward to show that $S_{\phi} > S_{\psi}$, or equivalently $S_{\psi} > 0$, necessitates that $\psi$ is a downside risk-averse transformation of $u$, but the converse, that $\psi'''' > 0$ implies $S_{\phi} > S_{\psi}$, is not true in the large, since the ranking by $\psi'''' > 0$ is not necessarily transitive but the ranking by $S_{\psi}$ is.

show, that $S_{\phi} > S_{\psi}$ is, instead, equivalent to a parametric increase in downside risk aversion. The restriction to parametric increases constrains the ranking by $\psi''''$ in precisely the manner needed to produce an ordering, by extending to the large the consistency of the latter in the small. For these parametric increases in downside risk aversion, where $S_{\phi}$ is increasing everywhere along a path from $u$ to $v$, comparative statics predictions obtained for small changes in risk preferences extend immediately to predictions for global changes, a property we exploit in establishing the predictions described above.

Parametric increases in downside risk aversion are formally introduced in the next section. In Section 3, we apply these results by extending the comparative statics predictions developed by Diamond and Stiglitz to increases in downside risk and in downside risk aversion. In Section 4, we analyze a thought experiment that cannot be examined at the second order concerning the choice of exposure to risk. Relationships with the literature are discussed in Section 5 and conclusions are offered in Section 6.

2. Parametric increases in downside risk aversion

As suggested in the introduction, it is important to recognize that the condition $S_{\phi} > 0$ differs from $\psi'''' > 0$ only for finite changes in risk preferences. For infinitesimal changes, the two conditions are one and the same. To investigate infinitesimal changes in risk preferences we first posit a parameterized family of transformations $\psi(u(y), c)$ of utility $u(y)$, giving rise to a family of utility functions $u(y, c) \equiv \psi(u(y), c)$. An infinitesimal increase in $c$ is represented by the transformation

$$\tilde{\psi}(u(y, c), \tilde{c}) = u(y, c + \tilde{c}),$$

so that $\tilde{\psi}/\tilde{c}|_{\tilde{c}=0}$ gives the effect on $u(y, c)$ of an infinitesimal increase in $c$.

Define $S_{\phi}(u(y), c)$ to be the Schwarzian for $\psi(u(y), c)$. The effects of an infinitesimal change in risk preferences on $S_{\phi}$, and on $\psi$ and its utility derivatives, are given by their partial derivatives with respect to $c$.

Lemma 1. An infinitesimal change in risk preferences has exactly the same effect on $S_{\phi}$ as on $\psi''''$.

Proof. Observe that, as a consequence of (3), we have $\tilde{\psi}'(u(y, c), 0) = 1$, $\tilde{\psi}''(u(y, c), 0) = 0$, and $\tilde{\psi}''''(u(y, c), 0) = 0$. Using these equations, straightforward differentiation of $S_{\phi} = \tilde{\psi}''''/\tilde{\psi} - 3/2(\tilde{\psi}'')^2$ with respect to $\tilde{c}$ yields

$$\frac{\partial S_{\phi}(u(y, c), \tilde{c})}{\partial \tilde{c}}|_{\tilde{c}=0} = \frac{\partial \tilde{\psi}''''(u(y, c), \tilde{c})}{\partial \tilde{c}}|_{\tilde{c}=0} = \frac{\partial \tilde{\psi}''''(u(y, c))}{\partial c}|_{c=0},$$

indicating that the effect of an infinitesimal change in preferences is the same for $S_{\phi}$ and $\psi''''$. Finally, observe that $\psi(u(y, c), 0) = y(u) = \psi(u(y, c)), c)$, which implies $S_{\phi}(u(y, c), 0) = S_{\phi}(u(y, c))$. Hence, $\tilde{\psi}''''(u(y, c))|_{\tilde{c}=0} = \tilde{\psi}''''(u(y, c), c)|_{c=0}$, so that the effect is the same for $S_{\phi}$ and $\psi''''$. □

Since $S_{\phi}(u(y, c), 0) = 0 = \tilde{\psi}''''(u(y, c), 0)$, Eq. (4) implies that an infinitesimal transformation of risk preferences is downside risk averse if and only if the Schwarzian of that transformation is positive. Therefore, in the small, the analogy between $S_{\phi} > 0$ and $R_{\phi} > 0$ is exact, as the latter is equivalent to $\psi'''' > 0$ and the former is equivalent to $\psi'''' > 0$. That is, $R_{\phi} > 0$ increases risk aversion and $S_{\phi} > 0$ increases downside risk aversion in the manner that accords with intuition. However, in the large, while risk-averse transformations remain equivalent to global increases in $R_{\phi}$, a downside risk-averse transformation continues to be implied by but no longer implies an increase in $S_{\phi}$. This asymmetry disappears, though, if, as is quite usual in an economic context, one looks at parametric families of changes in risk preferences. As shown in the Appendix, an increase in risk aversion between $u$ and $v$, in the sense of $R_{\phi} > 0$, is equivalent to $R_{\phi}$ increasing along a smooth path of utility functions between the two. As we shall see, exactly the same is true with regard to $S_{\phi} > 0$ and increases in the downside risk aversion measure $S_{\phi}$ along a path.

Definition 1. Utility $v(y)$ is parametrically more downside risk averse than $u(y)$ along a given smooth path (of utility functions) if given a parameterization of that path $u(y, c)$ for $c \in [0, c]$, so that $u(y) = u(y, 0)$ and $v(y) = u(y, c)$, we have $\tilde{\psi}''''(u(y, c), c)/\tilde{c}|_{c=0} > 0$ for all $c \in [0, c]$, with $\tilde{\psi}$ as given by (3).

This definition requires a positive value for the right-hand side of Eq. (4) everywhere along the path, indicating, by (2), that $S_{\phi}$ increases everywhere along the path. Defining $S_{\phi}(u(y, c))$ to be the Schwarzian for $u(y, c)$, we can write

$$\frac{\partial S_{\phi}(u(y, c))}{\partial c} > 0$$

as the condition for a parametric increase in downside risk aversion. We can now state the following theorem, where the proof of this and other results not in the body are found in Appendix A.

Theorem 1. Utility $v(y)$ is parametrically more downside risk averse than $u(y)$ along a given smooth path if and only if $S_{\phi}$ is uniformly increasing along the path from $u$ to $v$. Furthermore, $v(y)$ is parametrically more downside risk averse than $u(y)$ along the path if and only if any function that transforms one utility into another located later along the path is downside risk averse, be the change finite or infinitesimal.

By controlling infinitesimal changes in utility along a path from $u$ to $v$ so that each utility function along the path is a downside risk-averse transformation of its immediate predecessors, parametric increases in downside risk aversion yield rankings of utility functions by the magnitude of $S_{\phi}$, which, given the 1-cycle condition (2), provide orderings of utility. In order to speak of an ordering of utility functions alone, we will simply refer to utility $v(y)$ as
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