The increase in the resolvent energy of a graph due to the addition of a new edge

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ABSTRACT

The resolvent energy $E_R(G)$ of a graph $G$ on $n$ vertices whose adjacency matrix has eigenvalues $\lambda_1, \ldots, \lambda_n$ is the sum of the reciprocals of the numbers $n - \lambda_1, \ldots, n - \lambda_n$. We introduce the resolvent energy matrix $R(G)$ and present an algorithm that produces this matrix. This algorithm may also be used to update $R(G)$ when new edges are introduced to $G$. Using the resolvent energy matrix $R(G)$, we determine the increase in the resolvent energy $E_R(G)$ of $G$ caused by such edge additions made to $G$. Moreover, we express this increase in terms of the characteristic polynomial of $G$ and the characteristic polynomials of three vertex-deleted subgraphs of $G$.

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1. Introduction

Let $G$ be a simple graph on $n$ vertices having vertex set $V(G) = \{1, 2, 3, \ldots, n\}$ and edge set $E(G)$. Two vertices $u$ and $v$ are adjacent in $G$ if and only if $(u, v) \in E(G)$. If $(u, v) \notin E(G)$, then the graph $G + uv$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{(u, v)\}$. If $H$ has the same number of vertices as $G$, then $G$ is a proper subgraph of $H$ if $E(G) \subset E(H)$. The graph $G - u$ denotes the one-vertex-deleted subgraph of $G$ obtained from $G$ after removing vertex $u$ and the edges incident to $u$. The graph $G - u - v$ denotes the two-vertex-deleted subgraph $(G - u) - v$ of $G$.

Let $A$ be the $n \times n$ adjacency matrix of $G$. The graph $G$ has characteristic polynomial $\phi(G, x) = \det(xI - A)$, where $I$ is the identity matrix. The roots of $\phi(G, x)$ are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A$. The complete graph $K_n$ on $n$ vertices is the graph whose $n \times n$ adjacency matrix is $J - I$, where $J$ is the matrix of all ones. On the other hand, the empty graph $N_n$ on $n$ vertices is the graph whose adjacency matrix is the $n \times n$ zero matrix.

A walk of length $\ell$ in $G$ is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ of $G$ such that $(v_i, v_{i+1}) \in E(G)$ for all $i \in \{0, 1, 2, \ldots, \ell - 1\}$. A walk is closed if $v_0 = v_\ell$. The $k$th spectral moment $M_k(G)$ of $G$ is the sum of the $k$th powers of all of the eigenvalues of its adjacency matrix. Since $\text{tr}(M)$, the trace of a matrix $M$, is equal to the sum of the eigenvalues of $M$ [21], $M_k(G) = \text{tr}(A^k)$. Moreover, it is well known that the entry in the $j$th row and $k$th column of $A^\ell$ is equal to the number of walks of length $\ell$ in $G$, starting from $j \in V(G)$ and ending at $k \in V(G)$ [7]. Thus, $M_k(G)$ may be thought of as being the total number of closed walks of length $k$ in $G$, starting and ending at any vertex.

In 1978, Ivan Gutman, motivated by research on the total $\pi$-electron energy of molecules, defined the graph energy [15] as $\sum_{i=1}^{n} |\lambda_i|$. Starting from 2006, a surprisingly high number of graph energy variants were proposed in the literature, each with their own applications. This ‘energy deluge’ is discussed in reference [16], which additionally surveys and compares several of these graph energy variants. For a more thorough discussion of many such alternative graph energies, the reader

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is referred to the books [20,22]. Moreover, in a recent paper [24], new upper bounds were produced for several of these graph energies.

One of the more recent of these graph energy variants, the resolvent energy, was introduced in [19], following the earlier works by Estrada and Higham [12], and Chen and Qian [5]. It is defined by

$$ER(G) = \sum_{i=1}^{n} \frac{1}{n - \lambda_i}.$$  

Eventually, the resolvent energy was extensively studied [19,14,17,18]. Also, its Laplacian spectrum version was recently put forward [3,25].

In [19, Theorem 2], it was shown that

$$ER(G) = \sum_{k=0}^{\infty} \frac{M_k(G)}{n^{k+1}}.$$  

Thus, the resolvent energy belongs to a general class of cumulative vertex centrality measures based on closed walks, originally put forward by Estrada and Higham in [12]. This class contains graph invariants of the form

$$E(G) = \sum_{k=0}^{\infty} c_k M_k(G)$$  

with the sequence of positive real numbers $c_0, c_1, c_2, \ldots$ chosen such that the Maclaurin series $\sum_{k=0}^{\infty} c_k x^k$ converges to some function $f(x)$. Since $M_k(G) = \operatorname{tr}(A^k)$, we have the relation

$$E(G) = \operatorname{tr}(f(A)).$$

For instance, when $-n < x < n$, the series $\sum_{k=0}^{\infty} n^{-k+1} x^k$ converges to $(n-x)^{-1}$. Since the eigenvalues of $A$ also satisfy this inequality for any graph $G$ (see, for example, [26]), the summation $\sum_{k=0}^{\infty} n^{-k+1} A^k$ converges to $(nI - A)^{-1}$. Note that the eigenvalues of $(nI - A)^{-1}$ are $\frac{1}{n-\lambda_1}, \ldots, \frac{1}{n-\lambda_n}$, all of which are positive real numbers; hence, this inverse matrix exists for all graphs and is positive-definite. The resolvent energy $ER(G)$ is thus $E(G)$ with $c_k = \frac{1}{n^{k+1}}$ for all $k$ and with $f(x) = \frac{1}{n-x}$. The following lemma is consequently inferred.

**Lemma 1.1.** $ER(G) = \operatorname{tr}((nI - A)^{-1}).$

Two other particular cases of graph invariants pertaining to the class $E(G)$ of the form (1) are the Estrada index [4,8,10,11,13], in which

$$c_k = \frac{1}{k!} \text{ for all } k, \quad f(x) = e^x, \quad E(G) = EE(G) = \operatorname{tr}(e^A)$$

and the resolvent Estrada index [2,5,12] (defined for graphs that are not complete) in which

$$c_k = \frac{1}{(n-1)^k} \text{ for all } k, \quad f(x) = \frac{n-1}{n-1-x}.$$  

$$E(G) = EE_r(G) = (n-1) \operatorname{tr}((n-1I - A)^{-1}).$$

Clearly, there is a relation between the resolvent Estrada index $EE_r(G)$ and the resolvent energy $ER(G)$. Indeed, they are both based on the resolvent matrix of $A$, defined by $(zI - A)^{-1}$, where $z$ is a complex variable [28]. The resolvent matrix of $A$ exists for values of $z$ that are not eigenvalues of $A$.

It is clear, by Lemma 1.1, that studying the matrix $(nI - A)^{-1}$ should elucidate research on the resolvent energy. Because of this, we first establish strict bounds for the entries of the matrix $(nI - A)^{-1}$ in Section 2. Subsequently, we consider how this matrix changes after introducing a new edge to a graph $G$, leading to the algorithm in Section 4 that evaluates the resolvent energy of any graph without the need of evaluating any matrix inverse or any eigenvalues. In Section 5, the resolvent energy change $\delta$ caused by the introduction of a new edge in $G$ is quantified using entries of $(nI - A)^{-1}$. After deriving expressions for the entries of this matrix in terms of four characteristic polynomials related to $G$, we present a formula in Section 7 that evaluates $\delta$ from these characteristic polynomials.

2. The resolvent energy matrix

Motivated by the previous introductory section, we start this section by making the following definition.

**Definition 2.1.** The resolvent energy matrix of a graph $G$ on $n$ vertices having adjacency matrix $A$ is the matrix $R(G) = (nI - A)^{-1}$.

We denote the resolvent energy matrix $R(G)$ of Definition 2.1 by $R$ if the graph $G$ is clear from the context. Note that $R$ has rational entries, since it is the inverse of a matrix with integer entries. Because of this, $ER(G) = \operatorname{tr}(R(G))$ is a rational
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