Involution flows and discretization errors for nonlinear driftless control systems

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ABSTRACT

In a continuous-time nonlinear driftless control system, an involutive flow is a composition of input profiles that does not excite any Lie bracket. Such flow composition is trivial, as it corresponds to a "forth and back" cyclic motion obtained rewinding the system along the same path. The aim of this paper is to show that, on the contrary, when a (nonexact) discretization of the nonlinear driftless control system is steered along the same trivial input path, it produces a net motion, which is related to the gap between the discretization used and the exact discretization given by a Taylor expansion. These violations of involutivity can be used to provide an estimate of the local truncation error of numerical integration schemes. In the special case in which the state of the driftless control system admits a splitting into shape and phase variables, our result corresponds to saying that the geometric phases of the discretization need not obey an area rule, i.e., even zero-area cycles in shape space can lead to nontrivial geometric phases.

1. Introduction

For nonlinear control systems, it is well known that nonintegrability conditions on the vector fields are at the basis of our notions of (nonlinear) controllability and observability [1,2], as well as of many motion planning algorithms [2,3]. If the system is driftless, then nonintegrability of the vector fields (and Lie bracket conditions) allows to produce motion in directions not spanned by the vector fields, as in the parallel parking of a car [4–6].

Such directions can be excited by generating cyclic input motion, the "macroscopic" equivalent of a Lie bracket. In some special cases the area of such cycles can be taken as indicator of the net displacement along these Lie bracket generated directions. This is true for instance when a notion of geometric phase can be associated to the system. This happens for instance when the state of the system has the structure of principal fiber bundle, and the state vector can be split into shape variables, directly controlled by external inputs, and phase variables depending from the shape variables, with a cyclic change in the former inducing a net displacement on the latter. In continuous-time, geometric phases have been extensively studied in different fields, like classical mechanics [7], quantum mechanics [8,9], molecular systems, [10], robotics [2,3] and control theory [1]. For such systems, periodic inputs can be used to induce non-periodic movements in the phase variables, see for instance [9,11,12] for applications to swimming bodies in fluids, [13] for the falling cat problem, and [4–6] for the already mentioned parallel parking of a car. In these cases, the amplitude of the phase displacement is proportional to the area of the cyclic path in shape space. In particular, a zero-area cycle yields no geometric phase.

For general driftless control systems, the equivalent of a "zero-area" input cycle is an input trajectory that goes forth and back to the same point, "rewinding itself" along the same path. In continuous-time, in correspondence of such a trivial cyclic motion also the state vector returns to its starting point, since no Lie bracket is excited when accomplishing it. We say in this case that we have an involutive flow composition. The concept can be extended from driftless control systems to time-reversible control systems [14].

The aim of this paper is to show that the situation is different when the nonlinear system is discretized, in the sense that in discrete-time even an involutive flow composition may induce a net displacement in the state vector. We show that while an exact discretization obtained through a complete Taylor series expansion method does not violate involutivity, when instead the discrete-time system is obtained through an approximate discretization of a nonlinear system, for instance using an Euler method, then involutivity is violated. The displacement that is produced is related to the truncation error with respect to the exact discretization, and it appears to be both path-dependent and sampling length dependent. In particular, it tends to zero when the sampling interval tends to zero.
Although a large body of literature exists on determining discrete-time equivalents of the nonlinear notions used in control theory [15–20], in our knowledge, the observation that involutivity of flow compositions is lost when a system is discretized appears to be novel. Even in the deeply investigated context of geometric phases, the properties of discrete time geometric phases have never been investigated, let alone the existence of phase motions induced by zero-area shape cycles.

In the context of numerical integration of ODEs, if a non-exact discretization of the system driven by an external input is integrated along a path and then rewinded along the same path, an estimate of the numerical integration error is then produced, estimate that can be used to improve the integration routine itself. For driftless bilinear control systems, the numerical integration routine can be rendered exact. The observation that by decoupling involutive from non-involutive flows (i.e., those induced by Lie brackets) one can get an estimate on the fly of the local error of a non-involutive flow can still be obtained. Such input profile, given in (4), is a trivial one: a “forth and back” trajectory along the same path.

### Proposition 1
Consider the system (3) and the input protocol

\[
\begin{align*}
0 & \quad t < t_1 \\
\alpha_i & \quad t_1 \leq t < t_2 \\
0 & \quad t_2 \leq t < t_3 \\
-\alpha_i & \quad t_3 \leq t < t_4 \\
0 & \quad t \geq t_4
\end{align*}
\]

where \( t_1 \) and \( t_2 \) are begin and end of an input step, \( t_3 \) and \( t_4 \) are begin and end of an identical input step of opposite sign (i.e., \( t_2 - t_1 = t_4 - t_3 \)). We have the following:

**P1:** For the continuous-time system (3), the protocol (4) applied to all \( u_i \), \( i = 1, \ldots, m \), leads to \( x(t) = x(0) \) for \( t \geq t_4 \).

### Proof.
The flow of (3) under (4) is

\[
x(t_2) = \Phi(t_2 - t_1)^{\sum_{i=1}^{m} \alpha_i} (x(0))
\]

\[
x(t_4) = \Phi(t_4 - t_2)^{\sum_{i=1}^{m} \alpha_i} (x(t_2)) = \Phi(t_4 - t_2)^{\sum_{i=1}^{m} \alpha_i} (x(t_0))
\]

\[
= \Phi(t_4 - t_1 + t_3 - t_2)^{\sum_{i=1}^{m} \alpha_i} (x(0)) = x(0)
\]

and P1 is proven.

### 2. Involutivity of flows and discretization error for driftless nonlinear systems

#### 2.1. Involutivity of flows

Let \( f \) and \( g \) be \( C^\infty \) vector fields in \( \mathbb{R}^n \). Their Lie bracket is the vector field

\[
[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x).
\]

Denoting \( \Phi_t^f \) the flow at time \( t \) along the vector field \( f \) at \( x \) (a local diffeomorphism), and by \( \gamma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) the flow composition

\[
\gamma(t, x) = \Phi_t^f \circ \Phi_t^{-1} \circ \Phi_t^g \circ \Phi_t^{-1}(x) \tag{1}
\]

then for \( t \) sufficiently small we have that

\[
\gamma(t, x) = x + t^2 \left( \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x) \right) + O(t^3) \tag{2}
\]

i.e., the second order term of the expansion is given by the Lie bracket \([f, g]\). When the vector fields commute, i.e., when \([f, g] = 0\), then the right side of (2) is 0 and \( \gamma(4t, x) = x \) is the identity map.

When \([f, g](x) = 0 \forall x \in \mathbb{R}^n \), then a formula like (1) leads to \( \gamma(t, x) = x \) not only locally (i.e., for small \( t \)) but also for arbitrarily large \( t \). In the following we shall refer to this situation as *involutive flow composition*.

#### 2.2. Continuous-time involutive flows composition for driftless control systems

Consider the continuous-time nonlinear driftless system linear in the inputs:

\[
\dot{x} = \sum_{i=1}^{m} g_i(x)u_i. \tag{3}
\]

For (3), as soon as time-varying input profiles \( u_i(t) \) are chosen, a flow composition like (1) is normally not involutive. The following proposition shows that for a special choice of input profiles, involutive flows can still be obtained. Such input profile, given in (4), is a trivial one: a “forth and back” trajectory along the same path.

### Proposition 2
Consider the Euler discretization (5) and the exact discretization (6) of the system (3). Consider the input protocol

\[
\begin{align*}
0 & \quad k < k_1 \\
\alpha_i & \quad k_1 \leq k < k_2 \\
0 & \quad k_2 \leq k < k_3 \\
-\alpha_i & \quad k_3 \leq k < k_4 \\
0 & \quad k \geq k_4
\end{align*}
\]

where \( k_1, k_2, k_3, k_4 \) are begin and end of an input step, \( k_2 \) and \( k_3 \) are begin and end of an identical input step of opposite sign (i.e., \( k_2 - k_1 = k_4 - k_3 \)). We have the following:

**P2:** For the continuous-time system (3), the protocol (7) applied to all \( u_i \), \( i = 1, \ldots, m \), leads to \( x(t) = x(0) \) for \( t \geq t_4 \).

### Proof.

The Euler discretization of (3) is given by

\[
x(k + 1) = x(k) + h \sum_{i=1}^{m} g_i(x(k))u_i(k). \tag{5}
\]

For inputs \( u_i \) that are piecewise-constant in each sampling interval \( h \), an exact discretization of the system (3) is provided by a Taylor expansion [25]. It consists of the following infinite series

\[
x(k + 1) = x(k) + \sum_{j=1}^{\infty} B(j)(x(k), u(k)) \frac{h^j}{j!} \tag{6}
\]

where

\[
B^{(1)}(x, u) = \sum_{i=1}^{m} g_i(x)u_i \tag{7}
\]

\[
B^{(j+1)}(x, u) = \frac{\partial B^{(j)}(x, u)}{\partial x} \sum_{i=1}^{m} g_i(x)u_i. 
\]

The truncation error of the Euler discretization is then

\[
\epsilon(k) = \sum_{j=2}^{\infty} B^{(j)}(x(k), u(k)) \frac{h^j}{j!} \tag{8}
\]

From the expressions (5) and (6), it is clear that \( \lim_{h \to 0} \frac{x(k+h) - x(k)}{h} = \sum_{i=1}^{m} g_i(x(k))u_i(k) \) and \( \lim_{h \to 0} \epsilon(k) = 0 \).

Let us look at the equivalent of Proposition 2 for the two discretizations (5) and (6).

### Proposition 2
Consider the Euler discretization (5) and the exact discretization (6) of the system (3). Consider the input protocol
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