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An optimal investment strategy for a stream of liabilities generated by a step process in a financial market driven by a Lévy process

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ABSTRACT

In this paper we investigate an asset–liability management problem for a stream of liabilities written on liquid traded assets and non-traded sources of risk. We assume that the financial market consists of a risk-free asset and a risky asset which follows a geometric Lévy process. The non-tradeable factor (insurance risk or default risk) is driven by a step process with a stochastic intensity. Our framework allows us to consider financial risk, systematic and unsystematic insurance loss risk (including longevity risk), together with possible dependencies between them. An optimal investment strategy is derived by solving a quadratic optimization problem with a terminal objective and a running cost penalizing deviations of the insurer's wealth from a specified profit-solvency target. Techniques of backward stochastic differential equations and the weak property of predictable representation are applied to obtain the optimal asset allocation.

1. Introduction

We investigate an asset–liability management problem for a stream of liabilities written on liquid traded assets and non-traded sources of risk. In insurance business values of liabilities and timing of claims depend not only on non-market risk factors, like mortality or claim propensity, but are very often related to the financial market. Complicated structures of modern insurance products and non-trivial risks involved in running an insurance company call for a sound asset–liability management and derivation of hedging strategies.

In this paper we deal with a quadratic optimization or a quadratic hedging which is defined as the choice of an investment portfolio found by minimizing, in the mean square sense, an error of not fulfilling the contractual obligations. We consider a generalized penalty function, not only with a terminal objective, but also with a running cost. Such an optimization criterion has already been applied in insurance and finance in the context of asset allocations: see Detemple and Rindisbacher (2008), Gerrard et al. (2004) and Kohlmann and Peisl (2000). We believe that the inclusion of the running costs is very reasonable as it stabilizes the deviations of the wealth process over the whole investment period and forces the wealth process to meet the desired targets. A trajectory of the wealth process is subject to a penalty each time it does not fulfill the target. We can apply arbitrary targets, but the particular target we propose has a clear economic interpretation. It combines the insurer's preferences concerning the future expected profit and requirements on the capital, which has to be held over the contract duration, to cover the liability reserve. The result of our optimization problem, based on minimizing deviations with respect to the proposed target, is the asset allocation strategy generating wealth values which are as close as possible, in the mean square sense, to the levels under which the company earns the desired profit and fulfills the statutory reserve requirements.

We state and solve our quadratic optimization problem under the real-world measure/objective measure. In this respect our optimization problem is close to mean-variance Markowitz portfolio selection under which extreme deviations, profits and losses are penalized under the real-world expectations. Our formulation is fundamentally different from the quadratic hedging problem/risk minimization under a martingale measure (see Ankirchner and Imkeller, 2008; Dahl et al., 2008; Dahl and Møller, 2006; Møller, 2001) or local risk minimization (see Schweizer, 2008; Vandaele and Vanmaele, 2008) which requires

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a specification of a martingale measure (more or less arbitrary) or finding a proper martingale measure (which might be very difficult in some cases). As far as quadratic optimization under the real-world measure is concerned, there is a large literature on extensions of the Markowitz problem of minimizing variance given expected value or minimizing the mean square error, and we just cite the recent papers of Xie (2009) and Xie et al. (2008), where insurance liabilities follow, respectively, a Brownian motion and a Markovian regime switching Brownian motion, and where financial models based on a geometric Brownian motion and a Markovian regime switching Brownian motion are investigated. See as well Delong et al. (2008) where mean-variance hedging of an annuity under stochastic mortality is investigated in a financial model based on an exponential Lévy process.

We consider a financial market consisting of a risk-free asset, with deterministic return, and a risky asset whose price dynamics is described by an exponential Lévy process. It is well known, see Cont and Tankov (2004) and Kyprianou et al. (2005), that Lévy processes, unlike Brownian motions, can easily reproduce heavy tails, skewness and other distributional properties of assets' returns, and, what is very important, can generate discontinuities, sudden jumps, in price dynamics and capture price movements in a much better way. Nowadays, it is widely accepted that in the celebrated Black–Scholes model one should replace a Gaussian noise by a more general Lévy noise.

The liabilities which we investigate in this paper are related to the financial market and also depend, as is always the case for insurance products, on a non-tradeable factor. The payment process considered is similar to the payment processes from Dahl et al. (2008), Dahl and Møller (2006) and Møller (2001), but simultaneously takes into account equity-linked liabilities and unsystematic and systematic insurance risk. To the best of our knowledge, this is the first paper in the actuarial literature where an asset allocation strategy or a hedging strategy is derived for a stream of claims involving financial risk and unsystematic and systematic insurance risk. In our opinion integration of these three risk factors is very important for risk management. By unsystematic insurance risk we understand randomness of insurance claims (number of accidents or number of deaths) compared to expectation, and by systematic insurance risk we mean unpredictable changes in the underlying claim intensity (clients' claim frequency or population mortality intensity). The most important example of systematic insurance risk is the longevity risk. We recall that in Møller (2001) risk minimization hedging under a martingale measure and mean-variance optimization are considered for a stream of equity-linked liabilities, only under unsystematic insurance risk, in a continuous Black–Scholes financial model, whereas in Dahl et al. (2008) and Dahl and Møller (2006) risk minimization hedging under a martingale measure is considered only for traditional liabilities (without a financial component), under unsystematic and systematic mortality risk, in a financial model consisting of a bank account with a risk-free return driven by a CIR process, a bond and a mortality derivative. Finally in Vandaele and Vanmaele (2008) local risk minimization hedging is investigated for an equity-linked pure endowment and term insurance, only under unsystematic mortality risk, in a financial market based on an exponential Lévy process.

In this paper we deal with a loss process consisting of continuously paid benefits, a terminal benefit and randomly occurring benefits. Our payment process is based on a step process (or a random measure) with a stochastic intensity and a random transition probability, with possible dependencies between tradeable and non-tradeable factors, and goes far beyond the processes considered in Dahl et al. (2008), Dahl and Møller (2006), Møller (2001) or Vandaele and Vanmaele (2008). It can also include a regime switching dynamics of claims in the spirit of Xie (2009). We believe that our loss process should cover all interesting and most common real-life payment schemes. In taking a step process (a random measure) as claims' driving process we follow closely Becherer (2006), where exponential utility optimization with a terminal liability is investigated. We also mention another related paper by Ankirchner and Imkeller (2008), where an integrated financial and insurance model with correlated tradeable and non-tradeable risk factors driven by Brownian motions and a Lévy process is considered, which in some aspects is more general than ours but in many aspects less general, and a quadratic hedging problem with a terminal claim, stated under a martingale measure, is investigated. Finally, let us notice that our payment process can also describe claims from defaultable securities and one can apply our optimization problem to hedge defaultable instruments (see for example Bielecki and Jeanblanc, 2005).

In order to solve the stated optimization problem we apply techniques of backward stochastic differential equations and the property of weak predictable representation of local martingales. We point out that the approach in terms of Hamilton–Jacobi–Bellman equations is not applicable. BSDEs provide a powerful tool which allows handling many state variables and dependencies between them. As concerns mathematics, we follow Becherer (2006), Lim (2005), Øksendal and Hu (2008) and Møller (2001). The idea of solving our optimization problem is known but technically differs from the above papers and introduces new difficulties. We point out that, in contrast to our paper, Becherer (2006) and Øksendal and Hu (2008) do not provide any explicit solutions of backward stochastic differential equations. We first characterize the optimal investment strategy in terms of a solution of a backward stochastic differential equation driven by Brownian motions and random measures. Next, for two specific, but still very general, equity-linked payment processes with independent unsystematic and systematic insurance risk, we derive an explicit characterization. Finally, we mention that in Ankirchner and Imkeller (2008) a quadratic hedging strategy is characterized, not in the terms of a solution of the corresponding backward stochastic differential equation, but by Malliavin derivatives.

This paper is structured as follows. In Section 2 we introduce the financial model and the stream of liabilities. The optimization problem is stated in Section 3 and the solution is derived in Section 4. In Section 5 we consider two special cases of optimization under a martingale measure and the case of upward bounded jumps in the risky asset price dynamics. An explicit strategy for equity-linked life insurance and non-life insurance payment processes with independent unsystematic and systematic insurance loss risk is derived in Section 6, which also contains a numerical example. Section 7 contains some additional results and proofs omitted in the main text.

2. The model

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ and a finite time horizon $T < \infty$. We assume that $\mathbb{F}$ satisfies the usual hypotheses of completeness ($\mathcal{F}_0$ contains all sets of $\mathbb{P}$-measure zero) and right continuity ($\mathcal{F}_t = \mathcal{F}_{t+}$). The measure $\mathbb{P}$ is the real-world, objective probability measure and, if not specified, all expected values are taken with respect to $\mathbb{P}$. By $\mathcal{B}(A)$ we denote the Borel subsets of A, and by K a constant whose value may change from line to line.

In the following subsections we introduce the financial market and the payment process generated by the stream of liabilities.

2.1. The financial market

The financial market we consider consists of two tradeable instruments. The price of a risk-free asset $S_0 := (S_0(t), 0 \leq t \leq T)$