Where is matrix multiplication locally open?

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ABSTRACT

Let \((M_1, d_1), (M_2, d_2)\) be metric spaces. A map \(f : M_1 \rightarrow M_2\) is said to be \emph{locally open} at an \(x_1 \in M_1\), if for every \(\varepsilon > 0\) one finds a \(\delta > 0\) such that \(B(f(x_1), \delta) \subset f(B(x_1, \varepsilon))\); here \(B(x, r)\) stands for the closed ball with center \(x\) and radius \(r\).

We are particularly interested in the following special case: \(X, Y, Z\) are normed spaces, the spaces \(L(X, Y), L(Y, Z), L(X, Z)\) of linear continuous operators are provided with the operator norm, and the map under consideration is the bilinear map \((S, T) \mapsto S \circ T\) (from \(L(Y, Z) \times (L(X, Y)\) to \(L(X, Z))\). For which pairs \((S_0, T_0) \in L(Y, Z) \times (L(X, Y)\) is it locally open?

The main result of the paper gives a complete characterization of pairs \((S, T)\) at which this map is locally open in the case of finite-dimensional spaces \(X, Y, Z\).

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1. The problem

Let \(\mathcal{A}\) be a normed algebra and \((x_0, y_0) \in \mathcal{A}\). We say that \emph{multiplication is locally open} at \((x_0, y_0)\) if for every \(\varepsilon > 0\) one can find a \(\delta > 0\) such that

\[
\{x_0 \cdot y_0 + w \mid ||w|| \leq \delta\} \subset \{(x_0 + x) \cdot (y_0 + y) \mid ||x||, ||y|| \leq \varepsilon\}.
\]

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The case of commutative \( \mathcal{A} \) has been investigated in several papers (see [1–7]). Here we aim at characterizing the pairs \((x_0, y_0)\) where multiplication is locally open in the noncommutative normed algebra \( \mathcal{M}_n \) of real or complex \( n \times n \)-matrices for arbitrary \( n \in \mathbb{N} \).

In the framework of singularity theory the problem whether certain nonlinear maps are locally open at a given point has been studied for many decades. However, the interest in the special case of “natural” bilinear maps that occur in functional analysis is relatively new.

Whereas the “commutative” case (like pointwise multiplication in spaces of measurable or continuous functions) leads to problems in measure theory and topology the noncommutative matrix multiplication corresponds to a nonlinear map from \( 2n^2 \)- to \( n^2 \)-dimensional space, and there seem to be no general results that could be used here unless this mapping has full rank. We will show that rather elementary linear algebra leads to a complete characterization.

Another viewpoint is also possible. Let \( X, Y, Z \) be normed linear spaces and denote by \( L(X,Y), L(Y,Z) \) and \( L(X,Z) \) the spaces of continuous linear operators from \( X \) to \( Y \) etc. (These spaces are provided with the operator norm.) One can consider the bilinear map \( \Phi : L(Y,Z) \times L(X,Y) \to L(X,Z), (S, T) \mapsto S \circ T, \) and one may ask: at which pairs \((S_0, T_0)\) is \( \Phi \), the operator “multiplication”, locally open? The preceding case of normed algebras covers the situation \( X = Y = Z \), but what can be said in general? We provide a complete characterization for the case of finite dimensional \( X, Y, Z \) in section 2, and section 3 is concerned with the special case of multiplication in \( \mathcal{M}_n \).

2. Where is \((S, T) \mapsto S \circ T\) locally open?

First we collect some facts that concern arbitrary normed spaces \( X, Y, Z \) over the scalar field \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \). (We will drop the symbol “\( \circ \)” for the composition of maps.)

**Lemma 2.1.** Let \( S_0 \in L(Y, Z) \) and \( T_0 \in L(X, Y) \) be given.

(i) Suppose that \( T_0 \) has the following property: for every \( \varepsilon > 0 \) there exists \( T_\varepsilon \in L(X,Y) \) with \( ||T_\varepsilon|| \leq \varepsilon \) such that \( S_0 T_\varepsilon = 0 \), and \( T_0 + T_\varepsilon \) admits a left inverse (i.e., a \( \tilde{T} \in L(Y,X) \) with \( \tilde{T}(T_0+T_\varepsilon) = \text{Id}_X \)). Then multiplication is locally open at \((S_0, T_0)\).

(ii) If \( S_0 \) is such that for every \( \varepsilon > 0 \) there exists \( S_\varepsilon \in L(Y,Z) \) with \( ||S_\varepsilon|| \leq \varepsilon \) such that \( S_\varepsilon T_0 = 0 \) and \( S_0 + S_\varepsilon \) admits a right inverse (i.e., an \( \tilde{S} \in L(Z,Y) \) with \( (S_0 + S_\varepsilon) \tilde{S} = \text{Id}_Z \)), then multiplication is locally open at \((S_0, T_0)\).

(iii) If \( T_0 \) admits a left inverse or \( S_0 \) admits a right inverse, then multiplication is locally open at \((S_0, T_0)\).

(iv) Suppose that for every \( \varepsilon > 0 \) there exists \( T_\varepsilon \) with \( ||T_\varepsilon|| \leq \varepsilon \) (resp. \( S_\varepsilon \) with \( ||S_\varepsilon|| \leq \varepsilon \)) such that \( T_0 + T_\varepsilon \) has a left inverse (resp. \( S_0 + S_\varepsilon \) has a right inverse). Then multiplication is locally open at \((0, T_0)\) (resp. at \((S_0, 0)\)).
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