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Separability of stochastic production decisions from producer risk preferences in the presence of financial markets☆

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ABSTRACT

Separation results, as they are usually understood, refer to conditions under which a firm's production decisions are independent of its risk attitudes. Well-understood situations where separation occurs typically include those where technically feasible production opportunities are replicable in financial markets. This paper gives necessary and sufficient conditions for separation that go beyond these well-understood spanning conditions. To do so, we present a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions about the firm's technology and objective function.

Separation results, as they are usually understood, refer to conditions under which a firm's production decisions are independent of its risk attitudes. Well-understood situations where separation occurs typically include those where technically feasible production opportunities are replicable in financial markets (Magill and Quinzii, 1996). This phenomenon is usually referred to as 'spanning'.

Analysis of separation has focused on the derivation of 'spanning results'. The best-known such result is that of complete markets. Another well-known spanning result is that, with a single-output non-stochastic technology and stochastic prices, the production choices of an expected-utility maximizer are independent of its risk preferences if there exists an active forward market (Danthine, 1978; Holthausen, 1979; Anderson and Danthine, 1981, 1983a,b). A generalization of both of these spanning results occurs when a convex but stochastic production set lies completely within the span of existing financial assets (Magill and Quinzii, 1996; Milne, 1995).

The paper gives necessary and sufficient conditions for separation that go beyond these well-understood spanning conditions. To do so, we present a unified treatment of the production and financial decisions available to a firm facing frictionless financial markets and a stochastic production technology under minimal assumptions about the firm's technology and objective function.

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There are several reasons why such results are important. First, there is the potential analytical and empirical convenience that arises from being able to ignore a firm's idiosyncratic risk attitudes in studying and empirically modelling its production decisions.

Second, and perhaps more important in the long run, separation has important implications for asset valuation in incomplete markets. There are two dominant approaches to asset pricing: consumption-based pricing and arbitrage pricing. For assets within the span of the market, the two approaches coincide. But in the absence of perfect replicability, arbitrage pricing only places upper and lower bounds on asset prices. These bounds are frequently so imprecise as to be economically irrelevant (Bernardo and Ledoit, 2000; Cochrane, 2001). On the other hand, consumption-based pricing, while offering exactness for nonreplicable assets, lacks robustness, because results depend on potentially arbitrary assumptions on risk preferences.

The notions of idiosyncratic risk and real options based on nontradable securities suggests that economic agents typically must value assets that are not exactly replicable in financial markets. Thus, it is important to identify and to understand instances when exact pricing is available even in the absence of perfect replicability. If separation occurs, *sets of nonreplicable assets can be exactly priced using the market pricing kernel* without requiring arbitrary restrictions on preferences stronger than simple monotonicity. Thus, if separation occurs, the market-based pricing functional can be extended beyond the span of the market to encompass linear pricing of some nonreplicable assets. Separation conditions, therefore, provide an alternative to the noisy arbitrage-bound approach and the less robust consumption-based asset pricing approach.

In what follows, we first present our notation and some basic results from convex analysis. We then specify the firm's stochastic environment, both in terms of production opportunities and its access to financial markets. Next, following Chambers and Quiggin (2008), we introduce a "derivative-cost function" and briefly discuss its most relevant properties. After that we derive necessary and sufficient conditions for separation. Then the paper concludes.

1. Notation and preliminaries

For a convex function[1] $f : \Re^S \to \Re$, its *subdifferential* at $\mathbf{m}$ is the closed, convex set:

$$\partial f(\mathbf{m}) = \{\mathbf{k} \in \Re^S : \mathbf{k}'\mathbf{m} - f(\mathbf{m}) \geq \mathbf{k}'\hat{\mathbf{m}} - f(\hat{\mathbf{m}}) \text{ for all } \hat{\mathbf{m}}\}. \tag{1}$$

The elements of $\partial f(\mathbf{m})$ are referred to as subgradients. If f is differentiable at $\mathbf{m}$, $\partial f(\mathbf{m})$ is a singleton and corresponds to the usual gradient. Conversely, if $\partial f(\mathbf{m})$ is a singleton, f is differentiable at $\mathbf{m}$.

For f convex, its convex conjugate is denoted

$$f^*(\mathbf{k}) = \sup_{\mathbf{m}}\{\mathbf{k}'\mathbf{m} - f(\mathbf{m})\}.$$

If f is proper and closed,[2] then f^* is also a proper and closed convex function with

$$f(\mathbf{m}) = \sup_{\mathbf{k}}\{\mathbf{k}'\mathbf{m} - f^*(\mathbf{k})\}, \tag{2}$$

and on the relative interior of their domains

$$\mathbf{k} \in \partial f(\mathbf{m}) \Leftrightarrow \mathbf{m} \in \partial f^*(\mathbf{k}). \tag{3}$$

2. State-contingent technologies and the asset structure

We model a sole-proprietorship, price-taking firm facing a stochastic environment in a two-period setting. The current period, 0, is certain, but the future period, 1, is uncertain. Uncertainty is resolved by 'Nature' making a choice from $\Omega = \{1, 2, \ldots, S\}$. Each element of Ω is referred to as a state of nature. The only assumption on the firm's preferences is that they are increasing in period 0 consumption and period 1 consumption.

The firm's stochastic production technology is represented by a single-product, state-contingent input correspondence. Let $\mathbf{x} \in \Re_+^N$ be a vector of inputs committed prior to the resolution of uncertainty (period 0), and let $\mathbf{z} \in \Re_+^S$ be a vector of *ex ante* or state-contingent outputs also chosen in period 0 but realized in period 1. If state $s \in \Omega$ is realized (picked by 'Nature'), and the producer has chosen the *ex ante* input–output combination $(\mathbf{x}, \mathbf{z})$, then the realized or *ex post* output in period 1 is z_s. Because $\mathbf{z} \in \Re_+^S$ is a map from the state space, Ω, to the reals, it can be taken as a random variable.

The continuous input correspondence, $X : \Re_+^S \to 2^{\Re_+^N}$, maps state-contingent output vectors (random variables) into input sets that are capable of producing them:

$$X(\mathbf{z}) = \{\mathbf{x} \in \Re_+^N : \mathbf{x} \text{ can produce } \mathbf{z}\}.$$

[1] These results on convex functions are all drawn directly from Rockafellar (1970).

[2] f is proper if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x}$, and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x}$. A proper convex function is closed if it is lower-semicontinuous.