Insurance: Mathematics and Economics 42 (2008) 1128–1137

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A locally risk-minimizing hedging strategy for unit-linked life insurance contracts in a Lévy process financial market

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ARTICLE INFO

Article history:
Received October 2007
Received in revised form
March 2008
Accepted 4 March 2008

JEL classification:
G22
G11

MSC:
IB10
IE43
IM12

Keywords:
Unit-linked
Local risk-minimization
Hedging strategy
Lévy process

ABSTRACT

In [Riesner, M., 2006. Hedging life insurance contracts in a Lévy process financial market. Insurance Math. Econom. 38, 599–608] the (locally) risk-minimizing hedging strategy for unit-linked life insurance contracts is determined in an incomplete financial market driven by a Lévy process. The considered risky asset is not a martingale under the original measure and therefore, a change of measure to the minimal martingale measure is performed.

The goal of this paper is to show that the risk-minimizing hedging strategy under the new martingale measure which is found in the paper cited above is not the locally risk-minimizing strategy under the original measure. Finally, the real locally risk-minimizing strategy is derived and a relationship between the number of risky assets held in the proposed portfolio cited in the above-mentioned paper and the one proposed here is given.

1. Introduction

In Föllmer and Schweizer (1991) and in Schweizer (1990, 1993, 2001) the concept of locally risk-minimizing hedging strategies to hedge claims was introduced for the case that the discounted risky asset is a semimartingale. In Møller (1998) the hedging portfolio is constructed for unit-linked life insurance contracts with a pure endowment and a term insurance with single premium in the complete Black–Scholes market. The risky asset under consideration is a continuous semimartingale. In this Black–Scholes setting the locally risk-minimizing hedging strategy under the original measure is equivalent with the risk-minimizing hedging strategy under the minimal martingale measure. We emphasize that this equivalence is only proved for continuous semimartingales as it is the case in the Black–Scholes market.

In Riesner (2006) this theory of locally risk-minimizing hedging strategies is combined with the results of Møller (1998) but for a risky asset of which the price process is discontinuous as it follows a geometric Lévy process. However, determining the risk-minimizing hedging strategy under the minimal martingale measure does not provide the locally risk-minimizing hedging strategy in the Lévy case, as we will show in this paper. Moreover, we will derive the locally risk-minimizing hedging strategy by a direct construction of the Föllmer–Schweizer decomposition. Hereto, we explicitly check whether the risky asset satisfies all the necessary conditions in order to have the equivalence between the locally risk-minimizing hedging strategy and the Föllmer–Schweizer decomposition.

In the literature the equivalence is often applied in a wrong way. Therefore, we find it important to give an overview concerning ((pseudo) locally) risk-minimizing hedging strategies in Section 2. After repeating the setting of Riesner (2006) in Section 3, we then show in Section 4 that in Riesner (2006) the risk-minimizing hedging strategy under the new measure was found, but that this strategy is not the locally risk-minimizing one under the original measure. In Section 5 we show how to determine the real locally risk-minimizing hedging strategy under the original measure and we calculate the associated risk process. Finally in Section 6 we adapt the results of Riesner (2006) for unit-linked life insurance contracts with a pure endowment and a term insurance.

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0167-6687/$ – see front matter
doi:10.1016/j.insmatheco.2008.03.001

2. ((Pseudo) locally) risk-minimizing hedging strategy

In this section we recall the notions of the different risk-minimizing hedging strategies and point out the possible pitfalls.

2.1. Risk-minimizing hedging strategy

Föllmer and Sondermann (1986) introduced the concept of risk-minimizing hedging strategies for nonredundant contingent claims, written on a one-dimensional, square-integrable discounted risky asset X which is a martingale under the original measure P. We introduce the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ containing the information from the risky asset available up to time t. The goal is to minimize the variance of future costs: $R_t := E[(C_T - C_t)^2|\mathcal{F}_t]$ for all $t \in [0, T]$. Here C_t stands for the cost process and is defined as the difference between the value of the portfolio at time t and the gains made from trading in the financial market up to time t. It is important to notice that the risk-minimizing strategy penalizes losses and gains equally.

A trading strategy is of the form $\varphi = (\xi, \eta)$ with $\xi = (\xi_t)_{0\le t\le T}$ the number of risky assets and with $\eta = (\eta_t)_{0\le t\le T}$ the amount invested in the riskless asset. The value of the discounted portfolio at time t is then given by $V_t = \xi_t X_t + \eta_t$.

Definition 1 (*Strategy*). A couple $\varphi = (\xi, \eta)$ is called a strategy if

- ξ is a predictable process,
- $\xi \in L^2(X)$ with $L^2(X)$ the space of all $\mathbb{R}$-valued predictable processes ξ such that

$$\|\xi\|_{L^2(X)} := \left(E\left[\int_0^T \xi_u^2 \mathrm{d}[X, X]_u\right]\right)^{1/2} < \infty,$$

- η is adapted,
- $V = \xi X + \eta$ has right continuous paths and $E[V_t^2] < \infty$ for every $t \in [0, T]$ (i.e. $V_t \in L^2(P)$ for every $t \in [0, T]$).

Assume that the claim H is $\mathcal{F}_T$-measurable and square-integrable. A strategy is called H-admissible if the value of the portfolio at time T equals H.

Using the Galtchouk–Kunita–Watanabe decomposition (see Kunita and Watanabe (1967)) we know that the claim $H \in L^2(P)$ can be decomposed in the following way: $H = E[H] + \int_0^T \xi_s^* \mathrm{d}X_s + H^*$, with $\xi^* \in L^2(X)$, H^* a square-integrable P-martingale orthogonal to X with $H_0^* = 0$ P-almost surely.

The unique H-admissible risk-minimizing strategy $\hat{\varphi}$ is then given by

$$\hat{\varphi}_t = (\xi_t^*, E[H|\mathcal{F}_t] - \xi_t^* X_t) \quad \forall t \in [0, T].$$

An important property of risk-minimizing hedging strategies is the martingale property of the cost process $C_t(\hat{\varphi})$ defined by

$$C_t(\hat{\varphi}) = V_t(\hat{\varphi}) - \int_0^t \xi_u^* \mathrm{d}X_u. \tag{1}$$

The strategy is then called mean-self-financing.

Møller (1998) determined the risk-minimizing hedging strategy for unit-linked life insurance contracts with a claim payable at a fixed time only, while Møller (2001) dealt with general payment processes.

2.2. Locally risk-minimizing hedging strategy

We restrict us here to a one-dimensional discounted risky asset X which is no longer a martingale under the measure P, but only a semimartingale. Hence, we cannot any longer apply the theory of risk-minimization. To deal with such a case Schweizer (1993, 2001) introduced the concept of locally risk-minimizing strategy, where the conditional variances are kept as small as possible but now in a local manner.

The semimartingale X has the following decomposition

$$X = X_0 + Z + A, \tag{2}$$

with Z a square-integrable martingale for which $Z_0 = 0$, and with A a predictable process of finite variation $|A|$ (i.e. $\sup_\tau \sum_{i=1}^{N(\tau)} |A_{t_i} - A_{t_{i-1}}| < \infty$ for every partition τ of $[0, T]$).

We also need the following assumptions:

(A1) For P-almost all ω, the measure on $[0, T]$ induced by $\langle Z\rangle.(\omega)$ has the whole interval $[0, T]$ as its support. This means that $\langle Z\rangle$ should be P-almost surely strictly increasing on the whole interval $[0, T]$.
(A2) A is continuous.
(A3) A is absolutely continuous with respect to $\langle Z\rangle$ with a density α satisfying

$$E_Z[|\alpha| \log^+ |\alpha|] < \infty.$$

A sufficient condition is that $E[\langle \int \alpha \mathrm{d}Z\rangle] < \infty$.

To adapt the definition of a trading strategy in this case, we need that $\xi \in L^2(Z)$ and that $\int_0^T |\xi_u \mathrm{d}A_u| \in L^2(P)$. Then we are sure that $\int_0^t \xi_u \mathrm{d}X_u$, $0 \le t \le T$ is a semimartingale of class $\mathcal{S}^2$ meaning that

$$E\left[\int_0^T \xi_u^2 \mathrm{d}\langle Z\rangle_u + \left(\int_0^T |\xi_u \mathrm{d}A_u|\right)^2\right] < \infty.$$

The space formed by all the processes ξ satisfying this condition is given by Θ_S.

We first introduce the definition of a small perturbation.

Definition 2 (*Small Perturbation*). A trading strategy $\Delta = (\delta, \varepsilon)$ is called a small perturbation if it satisfies the following conditions:

- δ is bounded,
- $\int_0^T |\delta_u \mathrm{d}A_u|$ is bounded,
- $\delta_T = \varepsilon_T = 0$.

For any subinterval $(s, t]$ of $[0, T]$, we define the small perturbation

$$\Delta|_{(s,t]} := (\delta \mathbb{1}_{(s,t]}, \varepsilon \mathbb{1}_{[s,t)}).$$

Next, we define partitions $\tau = (t_i)_{0\le i\le N}$ of the interval $[0, T]$. A partition of $[0, T]$ is a finite set $\tau = \{t_0, t_1, \ldots, t_k\}$ of times with $0 = t_0 < t_1 < \cdots < t_k = T$ and the mesh size of τ is $|\tau| := \max_{t_i, t_{i+1}\in\tau}(t_{i+1} - t_i)$. A sequence $(\tau_n)_{n\in\mathbb{N}}$ is called increasing if $\tau_n \subseteq \tau_{n+1}$ for all n and it tends to the identity if $\lim_{n\to\infty} |\tau_n| = 0$.

Definition 3 (*Locally Risk-minimizing*). For a trading strategy φ, a small perturbation Δ and a partition τ of $[0, T]$ the risk quotient $r^\tau[\varphi, \Delta]$ is defined as follows:

$$r^\tau(\varphi, \Delta) := \sum_{t_i, t_{i+1}\in\tau} \frac{R_{t_i}(\varphi + \Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\varphi)}{E[\langle Z\rangle_{t_{i+1}} - \langle Z\rangle_{t_i}|\mathcal{F}_{t_i}]} \mathbb{1}_{(t_i,t_{i+1}]}. \tag{3}$$

A trading strategy φ is called locally risk-minimizing if $\liminf_{n\to\infty} r^{\tau_n}(\varphi, \Delta) \ge 0$ P_Z-a.e. on $\Omega \times [0, T]$ for every small perturbation Δ and every increasing sequence (τ_n) of partitions of $[0, T]$ tending to the identity.

Recently, it is proved that the condition (A1) is unnecessary by allowing an extension of the definition of the risk quotient (3) in Definition 3 (see Schweizer (in press)).

Lemma 1. *Assume that the semimartingale X, with the decomposition described in (2), satisfies (A1). If a trading strategy is locally risk-minimizing, then it is also mean-self-financing.*