Spatially heterogeneous Lotka–Volterra competition☆

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ABSTRACT

This paper studies the dynamics of the spatially-heterogeneous diffusive Lotka–Volterra competing species model. It focuses special attention in ascertaining the linear stability and multiplicity of the coexistence steady states. One of our main findings establishes that, as soon as any steady-state solution of the non-spatial model is linearly unstable somewhere in the inhabiting territory, Ω, any steady state of the spatial counterpart perturbing from it therein (as the diffusion rates, d_1, d_2, move away from 0) must be linearly unstable. From this general principle one can derive a number of rather astonishing consequences, as the multiplicity of the coexistence steady states when the non-spatial model exhibits founder control competition somewhere in Ω, say Ω_{bi}, even if Ω_{bi} is negligible empirically. Actually, this is the first available multiplicity result for small diffusion rates. Finally, based on a celebrated identity by M. Picone (1910), we are able to establish a new, rather striking, uniqueness result valid for general spatially heterogeneous models.

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1. Introduction

In this paper we consider the Lotka–Volterra competition reaction–diffusion heterogeneous system

$$\begin{cases} \dfrac{\partial u}{\partial t} + d_1 \mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv & \\ & \text{in } \Omega \times (0, +\infty), \\ \dfrac{\partial v}{\partial t} + d_2 \mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u(\cdot, 0) = u_0 > 0, \quad v(\cdot, 0) = v_0 > 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

as well as its associated elliptic counterpart

$$\begin{cases} d_1 \mathcal{L}_1 u = \lambda(x)u - a(x)u^2 - b(x)uv \\ d_2 \mathcal{L}_2 v = \mu(x)v - d(x)v^2 - c(x)uv & \text{in } \Omega, \\ \mathcal{B}_1 u = \mathcal{B}_2 v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

whose solutions are the steady states of the evolutionary model (1.1). In this model, Ω is a bounded domain of $\mathbb{R}^N$ with boundary, $\partial\Omega$, of class $\mathcal{C}^2$, and $\mathcal{L}_i$, $i = 1, 2$, are two self-adjoint uniformly elliptic operators in Ω of the type

$$\mathcal{L}_i = -\text{div}(A_i \nabla \cdot) + C_i, \qquad i = 1, 2, \tag{1.3}$$

with $A_i \in \mathcal{M}_N^{\text{sym}}(\mathcal{C}^2(\bar{\Omega}))$ and $C_i \in \mathcal{C}(\bar{\Omega})$. Given any Banach space, X, we are denoting by $\mathcal{M}_N(X)$ the set of matrices of order N with entries in X. Naturally, $\mathcal{M}_N^{\text{sym}}(X)$ stands for the subset of $\mathcal{M}_N(X)$ consisting of all the symmetric matrices. As far as concerns $\partial\Omega$, it is throughout assumed to be a $(N-1)$-dimensional manifold of class $\mathcal{C}^2$ consisting, for each $i \in \{1, 2\}$, of finitely many connected components of class $\mathcal{C}^2$

$$\Gamma_{\text{D}}^{i,j}, \quad \Gamma_{\text{R}}^{i,k}, \qquad 1 \le j \le n_{\text{D}}^i, \quad 1 \le k \le n_{\text{R}}^i,$$

for some integers $n_{\text{D}}^i, n_{\text{R}}^i \ge 0$. By the definition of component, they must be disjoint (see, e.g., J. Munkres [30]) and each of them must be, simultaneously, a relatively open and closed subset of $\partial\Omega$, because $\partial\Omega$ is a compact manifold without boundary. Some, or several, of these components might be empty, of course. We denote by

$$\Gamma_{\text{D}}^i = \bigcup_{j=1}^{n_{\text{D}}^i} \Gamma_{\text{D}}^{i,j}, \qquad \Gamma_{\text{R}}^i = \bigcup_{j=1}^{n_{\text{R}}^i} \Gamma_{\text{R}}^{i,j}, \qquad i = 1, 2,$$

the Dirichlet and Robin portions of

$$\partial\Omega = \Gamma_{\text{D}}^i \cup \Gamma_{\text{R}}^i, \quad i = 1, 2.$$

Associated with these decompositions of $\partial\Omega$, there are two boundary operators $\mathcal{B}_i$, $i = 1, 2$, defined by

$$\mathcal{B}_i h = \begin{cases} \mathcal{D}_i h := h & \text{in } \Gamma_{\text{D}}^i, \\ \mathcal{R}_i h := \langle \mathbf{n}, A_i \nabla h \rangle + \beta_i h & \text{in } \Gamma_{\text{R}}^i, \end{cases} \quad \text{for every } h \in W^{2,p}(\Omega), p > N, \tag{1.4}$$

where $\beta_i \in \mathcal{C}(\partial\Omega)$ and $\mathbf{n}$ stands for the outward normal vector field along $\partial\Omega$. Thus, for each $i = 1, 2$, Γ_{D}^i and Γ_{R}^i are the portions of the edges of the inhabiting territory, $\partial\Omega$, where the corresponding species, u, or v, obeys a boundary condition of Dirichlet ($\mathcal{D}$) or Robin ($\mathcal{R}$) type, respectively. In particular, we may denote $\mathcal{B}_i = \mathcal{D}$ when $\Gamma_{\text{D}}^i = \partial\Omega$. In most of this paper, we are assuming that $\lambda, \mu, a, b, c, d \in \mathcal{C}(\bar{\Omega})$ satisfy

$$b(x) > 0 \text{ and } c(x) > 0 \text{ for all } x \in \Omega, \qquad \min_{\bar{\Omega}} a > 0, \qquad \min_{\bar{\Omega}} d > 0, \tag{1.5}$$

though in Sections 2–4 the hypothesis on b and c can be relaxed to $b, c \ge 0$ in $\bar{\Omega}$.

Throughout this paper, for any given function $h \in \mathcal{C}(\Omega)$, we shall denote $h_+ := \max\{h, 0\}$. It is said that h is positive, $h > 0$ or $h \gneq 0$ (in Ω), if $h \ge 0$ with $h \ne 0$. Also, for any given $h \in \mathcal{C}^1(\bar{\Omega})$, it is said that h is strongly positive (in Ω), $h \gg 0$, if it satisfies

$$h(x) > 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad \frac{\partial h}{\partial \mathbf{n}}(x) := \langle \mathbf{n}(x), \nabla h(x) \rangle < 0 \quad \text{for all } x \in h^{-1}(0) \cap \partial\Omega.$$

Except for the general existence results of Chapter 7 of [25], most of the available literature on Lotka–Volterra competing species models dealt with the very special cases when either $\Gamma_{\text{R}}^1 = \Gamma_{\text{R}}^2 = \emptyset$, or