Capital allocation for portfolios with non-linear risk aggregation

Tim J. Boonen a, *, Andreas Tsanakas b, Mario V. Wüthrich c,1

a Amsterdam School of Economics, University of Amsterdam, The Netherlands
b Faculty of Actuarial Science and Insurance, Cass Business School, City University London, United Kingdom
c RiskLab, Department of Mathematics, ETH Zurich, Switzerland

A R T I C L E I N F O

Article history:
Received June 2016
Received in revised form September 2016
Accepted 13 November 2016
Available online 21 November 2016

JEL classification:
C71
G22
Keywords:
Capital allocation
Euler rule
Fuzzy core
Aumann–Shapley value
Risk measures

A B S T R A C T

Existing risk capital allocation methods, such as the Euler rule, work under the explicit assumption that portfolios are formed as linear combinations of random loss/profit variables, with the firm being able to choose the portfolio weights. This assumption is unrealistic in an insurance context, where arbitrary scaling of risks is generally not possible. Here, we model risks as being partially generated by Lévy processes, capturing the non-linear aggregation of risk. The model leads to non-homogeneous fuzzy games, for which the Euler rule is not applicable. For such games, we seek capital allocations that are in the core, that is, do not provide incentives for splitting portfolios. We show that the Euler rule of an auxiliary linearised fuzzy game (non-uniquely) satisfies the core property and, thus, provides a plausible and easily implemented capital allocation. In contrast, the Aumann–Shapley allocation does not generally belong to the core. For the non-homogeneous fuzzy games studied, Tasche’s (1999) criterion of suitability for performance measurement is adapted and it is shown that the proposed allocation method gives appropriate signals for improving the portfolio underwriting profit.

© 2016 Elsevier B.V. All rights reserved.

1. Introduction

Financial firms often carry out a process of capital allocation, whereby the firm’s total capital requirement is apportioned to different lines of business and sub-portfolios. The total capital is typically calculated using a risk measure, such as standard deviation, Tail-Value-at-Risk (TVaR) or Value-at-Risk (VaR), and reflects the diversification from risk pooling in the portfolio. Alternative allocation methods reflect the different ways in which individual risks and sub-portfolios contribute to the total capital. There are several streams in the literature, respectively motivated by arguments from: (a) cooperative game theory (Denault, 2001; Tsanakas and Barnett, 2003; Kalkbrener, 2005; Hougaard and Smligins, 2016; Csóka et al., 2009); (b) performance and portfolio management (Tasche, 1999; Buch et al., 2011); (c) market valuation of assets and liabilities (Myers and Read, 2001; Zanjani, 2010; Sherris, 2006; Bauer and Zanjani, 2015); and (d) optimisation (Dhaene et al., 2003, 2012).

A standard assumption in the literature is that portfolios are formed as linear combinations of random loss/profit variables, with the decision maker being able to choose the portfolio weights. As already noted by Mildenhall (2004, 2006), this assumption is not necessarily appropriate in an insurance context. Losses from an insurance portfolio arise from the aggregation of claims that are generally not perfectly dependent. Increasing the exposure in a line of business within an insurance portfolio does not correspond to linearly scaling up the loss, but to adding more policies to the portfolio (a similar lack of linear scalability is observed in credit risk portfolios). When insurance policies are independent, then claims can be modelled via Lévy processes; for instance the compound Poisson process is the canonical example in the actuarial literature. The risk capital is then determined by a risk measure evaluated at the aggregate claim. While the risk measure typically used is positively homogeneous (e.g. the standard deviation or a distortion risk measure such as TVaR), the risk capital is not homogeneous in the exposures; due to diversification effects, doubling the number of insurance policies written does not lead to a doubling of the required risk capital.

In this paper we address capital allocation using a model that incorporates both Lévy and linear portfolio components. Our main focus is on game theoretical arguments. A function $r_t$,
mapping exposures to capital requirements, is called a fuzzy game. A fundamental question in this framework is whether candidate capital allocations belong to the core of \( r \). An allocation that belongs to the core of \( r \) ensures that a lower amount of capital is allocated to any sub-portfolio, compared to it being operated on a stand-alone basis. When portfolios are linear, scalable, then the gradient of the fuzzy game, known as the Euler rule (Tasche, 1999), provides the unique core allocation (Aubin, 1979, 1981; Denault, 2001). However, in our case, as the fuzzy game \( r \) is not homogeneous, the Euler rule is no longer applicable.

We introduce an auxiliary homogeneous fuzzy game \( \tilde{r} \), which can be seen as a linearisation of the original fuzzy game \( r \). The values of \( r \) and \( \tilde{r} \) coincide in the cases of full/no participation in individual lines of business. Furthermore, it is shown that, for risk models such as TVaR or standard deviation, which preserve convex order, \( \tilde{r} \) is dominated by \( r \). As a consequence the Euler rule for the auxiliary fuzzy game \( \tilde{r} \) belongs to the core of \( r \). However, we note that for risk measures like VaR, which do not preserve convex order, the core may be empty.

Thus, our method gives a general construction of core capital allocations, applicable to insurance portfolios. Our finding is particularly relevant for the practice of insurance risk management. The Euler rule is often applied in the insurance industry, with the implicit but incorrect assumption of portfolio linearity. Our results show that using such a ‘wrong’ model with homogeneous risks turns out to give a risk allocation that is a core element of the ‘correct’ underlying fuzzy game.

Our proposed capital allocation method improves upon previous attempts to deal with risk portfolios that are non-linear in the exposures. In particular, Powers (2007) studies the Aumann–Shapley value (Aumann and Shapley, 1974), which is applicable in the case of non-homogeneous fuzzy games. However, the Aumann–Shapley value typically fails to produce computationally tractable risk capital allocations. Furthermore, we show the Aumann–Shapley value does not need to be in the core of \( r \), except in the special case where the fuzzy game \( r \) is concave. Therefore, Aumann–Shapley allocations can produce incentives for portfolio fragmentation.

Finally, we consider implications for portfolio management. For linear portfolios, it is possible to derive appropriate signals for portfolio management, by evaluating for each line of business the return on allocated capital, as calculated by the Euler rule (Tasche, 1999). However, the diversification implicit in aggregating insurance risks necessitates the consideration of capital constraints. We adapt the arguments of Tasche (1999) and show that the proposed capital allocation method provides appropriate signals for increasing the aggregate underwriting profit.

Section 2 introduces the model and risk measures used. Section 3 contains the main contributions of this paper, including the proposed capital allocation method and the study of the core of the fuzzy game \( r \). Signals for portfolio management are discussed in Section 4 and brief conclusions given in Section 5.

2. Model outline

We consider a filtered probability space \((\mathcal{G}, \mathcal{F}, P, \{\mathcal{F}_w\}_{0 \leq w \leq W})\) with \( \mathcal{F} = \mathcal{F}_0 \) for given \( W > 1 \). Throughout, (in-)equalities between random variables are understood in the \( P \)-a.s. sense.

A financial firm, such as an insurance company, writes \( I \) lines of business. The exposure of the financial firm to the \( i \)th line of business is described by \( 0 \leq w_i \leq W \), and the loss arising from that line of business is denoted by the random variable \( X_i(w_i) \) for \( i = 1, \ldots, I \). The total loss of the financial firm as a function of the exposure \( \mathbf{w} = (w_1, \ldots, w_I) \) is denoted by

\[
S(\mathbf{w}) = \sum_{i=1}^{I} X_i(w_i). \tag{1}
\]

The current (base-line) exposure of the firm is \( \mathbf{w} = \mathbf{1} \), leading to the total loss \( S(\mathbf{1}) = \sum_{i=1}^{I} X_i(1) \).

To provide a tractable structure for the ways in which changes in exposure \( \mathbf{w} \) affect the joint law of the losses, we introduce the following model. Consider the random vector \( \mathbf{Y}(\mathbf{w}) = (Y_i(w_1), \ldots, Y_i(w_I)) \), where \( Y_i = Y_i(w_1) \) are \( \mathcal{F}_w \)-adapted independent increasing Lévy processes for \( i = 1, \ldots, I \). Let \( \mathbf{Z} = (Z_1, \ldots, Z_I) \) be an \( \mathcal{F}_W \)-measurable random vector (having possibly dependent components), and assume that \( \mathbf{Z} \) and \( (Y_i)_{i=1,\ldots,I} \) are independent. Then, we define for \( i = 1, \ldots, I \) and exposure \( w_i \in [0, W] \) the loss arising from the \( i \)th line of business by

\[
X_i(w_i) = Y_i(w_i) + w_i Z_i. \tag{2}
\]

We also write \( X_i = (X_i(w))_{0 \leq w \leq W} \) and \( \mathbf{Y}(\mathbf{w}) = \mathbf{Y}(\mathbf{w}) + \mathbf{w} \cdot \mathbf{Z} \), where \( \mathbf{w} \cdot \mathbf{Z} \) is the Hadamard (element-wise) product.

We stress that the ‘development’ of the stochastic processes \( \mathbf{Y} \), does not represent elapsed time, but increase in exposure. Thus, stopping the process at point \( w_i \) corresponds to placing a limit on the exposure of the \( i \)th line of business. Losses from insurance portfolios can be modelled as aggregations of (typically independent) claim amounts from different policies, for which Lévy processes (with their connection to infinitely divisible distributions) provide an appropriate representation. Henceforth, the dynamics of the processes \( \mathbf{Y} \) are of particular interest in this paper; all distributions and moments evaluated are with respect to information \( \mathcal{F}_0 \).

The model allows two special cases:

- If \( Y_i(w_i) = 0 \) for all \( i = 1, \ldots, I \), then \( X_i(w_i) = w_i X_i(1) \), and the loss of the \( i \)th line of business scales linearly in \( w_i \). This is the common situation described e.g. by Tasche (1999), where \( X_i(1) \) can be seen as the (negative) values of tradable assets and \( w_i \) are portfolio weights.

- If \( Z_i = 0 \) for all \( i = 1, \ldots, I \), then the line of business losses \( X_i \) are Lévy processes. In that case, \( X_i \) can represent a standard actuarial risk model. For example, if \( X_i \) is a Poisson process with unit intensity, then \( X_i(w_i) \sim \text{Poisson}(w_i) \). Now, \( X_i(w_i) \) is no longer linearly scalable with exposure, in particular, we have for its variance the property \( \mathbb{V} (X_i(w_i)) = w_i X_i(1) \).

The full model \( (2) \) can then be viewed as an insurance risk component \( Y_i(w_i) \) augmented by common shocks \( \mathbf{Z} \) that simultaneously affect all claims from the \( i \)th line of business. Dependence between losses of different lines of business is induced by the possible dependence between the elements of \( \mathbf{Z} \); it is straightforward to see that if \( i \neq j \) we have covariance \( \mathbb{C}(X_i(w_i), X_j(w_j)) = w_i w_j C(Z_i, Z_j) \).

In general, the processes \( X_i \) do not have independent increments; however the increments remain identically distributed.

**Lemma 2.1.** Let \( 0 \leq v_i < w_i \leq W \) for all \( i = 1, \ldots, I \). Then, \( \mathbf{X}(\mathbf{w}) - \mathbf{X}(\mathbf{v}) \overset{d}{=} \mathbf{X}(\mathbf{w} - \mathbf{v}) \).

**Proof.** The claim follows from the following identity:

\[
\mathbf{X}(\mathbf{w}) - \mathbf{X}(\mathbf{v}) = \mathbf{Y}(\mathbf{w}) + \mathbf{w} \cdot \mathbf{Z} - (\mathbf{Y}(\mathbf{v}) + \mathbf{v} \cdot \mathbf{Z})
\]

\[
\overset{d}{=} \mathbf{Y}(\mathbf{w} - \mathbf{v}) + (\mathbf{w} - \mathbf{v}) \cdot \mathbf{Z}
\]

\[
= \mathbf{X}(\mathbf{w} - \mathbf{v}).
\]

\(2\) \( X \) is defined on the domain \([0, W]^I \) with \( W > 1 \), which extends the exposures beyond the base-line exposure \( 1 \). In part this is due to mathematical convenience, since in Section 3 derivatives of functions (fuzzy games) at \( \mathbf{w} = 1 \) will be taken. But also, in Section 4, the strategic behaviour of the firm is considered, which includes the potential for portfolio expansion beyond base-line exposure \( \mathbf{w} = 1 \).

\(3\) An alternative interpretation of model \( (2) \) arises by considering \( \mathbf{z} \), as losses from a linear portfolio with weights \( \mathbf{w} \). Then, the Lévy component \( Y_i \) can be seen as representing operational costs associated with the \( i \)th line of business. In this interpretation, expected operational costs are increasing linearly with exposure, but become less volatile as the portfolio grows.
دریافت فوری متن کامل مقاله

امکان دانلود نسخه تمام متن مقالات انگلیسی
امکان دانلود نسخه ترجمه شده مقالات
پذیرش سفارش ترجمه تخصصی
امکان جستجو در آرشیو جامعی از صدها موضوع و هزاران مقاله
امکان دانلود رایگان ۲ صفحه اول هر مقاله
امکان پرداخت اینترنتی با کلیه کارت های عضو شتاب
دانلود فوری مقاله پس از پرداخت آنلاین
پشتیبانی کامل خرید با بهره مندی از سیستم هوشمند رهگیری سفارشات