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The affine arbitrage-free class of Nelson–Siegel term structure models

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ABSTRACT

We derive the class of affine arbitrage-free dynamic term structure models that approximate the widely used Nelson–Siegel yield curve specification. These arbitrage-free Nelson–Siegel (AFNS) models can be expressed as slightly restricted versions of the canonical representation of the three-factor affine arbitrage-free model. Imposing the Nelson–Siegel structure on the canonical model greatly facilitates estimation and can improve predictive performance. In the future, AFNS models appear likely to be a useful workhorse representation for term structure research.

1. Introduction

Understanding the dynamic evolution of the yield curve is important for many tasks, including pricing financial assets and their derivatives, managing financial risk, allocating portfolios, structuring fiscal debt, conducting monetary policy, and valuing capital goods. To investigate yield curve dynamics, researchers have produced a vast literature with a wide variety of models. However, those models tend to be either theoretically rigorous but empirically disappointing, or empirically successful but theoretically lacking. In this paper, we introduce a theoretically rigorous yield curve model that simultaneously displays empirical tractability, good fit, and good forecasting performance.

Because bonds trade in deep and well-organized markets, the theoretical restrictions that eliminate opportunities for riskless arbitrage across maturities and over time hold powerful appeal, and they provide the foundation for a large finance literature on arbitrage-free (AF) models that started with Vasiček (1977) and Cox et al. (1985). Those models specify the risk-neutral evolution of the underlying yield curve factors as well as the dynamics of risk premia. Following Duffie and Kan (1996), the affine versions of those models are particularly popular, because yields are convenient linear functions of underlying latent factors (state variables that are unobserved by the econometrician) with parameters, or "factor loadings", that can be calculated from a simple system of differential equations.

Unfortunately, the canonical affine AF models often exhibit poor empirical time series performance, especially when forecasting future yields (Duffee, 2002). In addition, and crucially, the estimation of those models is known to be problematic, in large part because of the existence of numerous likelihood maxima that have essentially identical fit to the data but very different implications for economic behavior. The empirical problems appear to reflect an underlying model over-parameterization, and as a solution, many researchers (e.g. Duffee, 2002; Dai and Singleton, 2002) simply restrict to zero those parameters with small t-statistics in a first round of estimation. The resulting more parsimonious structure is typically somewhat easier to estimate and has fewer troublesome likelihood maxima. However, the additional restrictions on model structure are not well motivated theoretically or statistically, and their arbitrary application and the computational burden of estimation effectively preclude robust model validation and thorough simulation studies of the finite-sample properties of the estimators.

In part to overcome the problems with empirical implementation of the canonical affine AF model, we develop in this paper a new class of affine AF models based on the workhorse yield curve representation introduced by Nelson and Siegel (1987) and extended to dynamic environments by Diebold and Li (2006). (We refer to the Diebold–Li extension as dynamic Nelson–Siegel, or DNS.) Thus, from one perspective, we take the theoretically rigorous but empirically problematic affine AF model and make it empirically tractable by incorporating DNS elements.

From an alternative perspective, we take the DNS model, which is empirically successful but theoretically lacking, and make it

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rigorous by imposing absence of arbitrage. This rigor is important because the Nelson–Siegel model is extremely popular in practice, among both financial market practitioners and central banks (e.g., Svensson, 1995; Bank for International Settlements, 2005; Gürkaynak et al., 2007). DNS's popularity stems from several sources, both empirical and theoretical, as discussed in Diebold and Li (2006). Empirically, the DNS model is simple and stable to estimate, and it is quite flexible and fits both the cross-section and time series of yields remarkably well, in many countries and periods, and for many grades of bonds. Theoretically, DNS imposes certain economically desirable properties, such as requiring the discount function to approach zero with maturity, and Diebold and Li (2006) show that it corresponds to a modern three-factor model of time-varying level, slope and curvature. However, despite its good empirical performance and a certain amount of theoretical appeal, DNS fails on an important theoretical dimension: it does not impose the restrictions necessary to eliminate opportunities for riskless arbitrage (e.g. Filipović, 1999; Diebold et al., 2005). This motivates us in this paper to introduce the class of AF Nelson–Siegel (AFNS) models, which are affine AF term structure models that maintain the DNS factor loading structure.

In short, the AFNS models proposed here combine the best of the AF and DNS traditions. Approached from the AF side, they maintain the AF theoretical restrictions of the canonical affine models but can be easily estimated, because the dynamic Nelson–Siegel structure helps to identify the latent yield curve factors and delivers analytical solutions (which we provide) for zero-coupon bond prices. Approached from the DNS side, they maintain the simplicity and empirical tractability of the popular DNS models, while simultaneously enforcing the theoretically desirable property of absence of riskless arbitrage.

After deriving the new class of AFNS models, we examine their in-sample fit and out-of-sample forecast performance relative to standard DNS models. For both the DNS and the AFNS models, we estimate parsimonious and flexible versions (with both independent factors and more richly parameterized correlated factors). We find that the flexible versions of both models are preferred for in-sample fit, but that the parsimonious versions exhibit significantly better out-of-sample forecast performance. As a final comparison, we also show that an AFNS model can outperform the canonical affine AF model in forecasting.

We proceed as follows. First we present the main theoretical results of the paper; in Section 2 we derive the AFNS class of models, and in Section 3 we characterize the precise relationship between the AFNS class and the canonical representation of affine AF models. We next provide an empirical analysis of four leading DNS and AFNS models, incorporating both parsimonious and flexible versions; in Section 4 we examine in-sample fit, and in Section 5 we examine out-of-sample forecasting performance. We conclude in Section 6, and we provide proofs and additional technical details in several appendices.

2. Nelson–Siegel term structure models

Here we review the DNS model and introduce the AFNS class of AF affine term structure models that maintain the Nelson–Siegel factor loading structure.

2.1. The dynamic Nelson–Siegel model

The original Nelson–Siegel model fits the yield curve with the simple functional form

$$y(\tau) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} \right) + \beta_2 \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right), \quad (1)$$

where $y(\tau)$ is the zero-coupon yield with τ months to maturity, and $\beta_0, \beta_1, \beta_2$, and λ are parameters.

As noted earlier, this representation is commonly used by central banks and financial market practitioners to fit the cross-section of yields. Although such a static representation is useful for some purposes, a dynamic version is required to understand the evolution of the bond market over time. Hence Diebold and Li (2006) suggest allowing the β coefficients to vary over time, in which case, given their Nelson–Siegel loadings, the coefficients may be interpreted as time-varying level, slope and curvature factors. To emphasize this, we re-write the model as

$$y_t(\tau) = L_t + S_t \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} \right) + C_t \left(\frac{1 - e^{-\lambda \tau}}{\lambda \tau} - e^{-\lambda \tau} \right). \quad (2)$$

Diebold and Li assume an autoregressive structure for the factors, which produces the DNS model, a fully dynamic Nelson–Siegel specification. Indeed, it is a state-space model, with the yield factors as state variables, as emphasized in Diebold et al. (2006).

Empirically, the DNS model is highly tractable and typically fits well. Theoretically, however, it does not require that the dynamic evolution of yields cohere such that arbitrage opportunities are precluded. Indeed, the results of Filipović (1999) imply that *whatever* stochastic dynamics are chosen for the DNS factors, it is impossible to preclude arbitrage at the bond prices implicit in the resulting Nelson–Siegel yield curve. In the next subsection, we show how to remedy this theoretical weakness.

2.2. The arbitrage-free Nelson–Siegel model

Our derivation of the AFNS model starts from the standard continuous-time affine AF structure of Duffie and Kan (1996).[1] To represent an affine diffusion process, define a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, where the filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$ satisfies the usual conditions (Williams, 1997). The state variable X_t is assumed to be a Markov process defined on a set $M \subset \mathbf{R}^n$ that solves the stochastic differential equation (SDE),[2]

$$dX_t = K^Q(t)[\theta^Q(t) - X_t]dt + \Sigma(t)D(X_t, t)dW_t^Q, \quad (3)$$

where W^Q is a standard Brownian motion in $\mathbf{R}^n$, the information of which is contained in the filtration $(\mathcal{F}_t)$. The drifts and dynamics $\theta^Q : [0, T] \to \mathbf{R}^n$ and $K^Q : [0, T] \to \mathbf{R}^{n\times n}$ are bounded, continuous functions.[3] Similarly, the volatility matrix $\Sigma : [0, T] \to \mathbf{R}^{n\times n}$ is a bounded, continuous function, while $D : M \times [0, T] \to \mathbf{R}^{n\times n}$ has a diagonal structure with ith diagonal entry given by $\sqrt{\gamma^i(t) + \delta_1^i(t)x_t^1 + \cdots + \delta_n^i(t)x_t^n}$.

To simplify the notation, $\gamma(t)$ and $\delta(t)$ are defined as

$$\gamma(t) = \begin{pmatrix} \gamma^1(t) \\ \vdots \\ \gamma^n(t) \end{pmatrix} \quad \text{and} \quad \delta(t) = \begin{pmatrix} \delta_1^1(t) & \cdots & \delta_n^1(t) \\ \vdots & \ddots & \vdots \\ \delta_1^n(t) & \cdots & \delta_n^n(t) \end{pmatrix},$$

where $\gamma : [0, T] \to \mathbf{R}^n$ and $\delta : [0, T] \to \mathbf{R}^{n\times n}$ are bounded, continuous functions. Given this notation, the SDE of the state variables can be written as

$$dX_t = K^Q(t)[\theta^Q(t) - X_t]dt + \Sigma(t) \times \begin{pmatrix} \sqrt{\gamma^1(t) + \delta^1(t)X_t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{\gamma^n(t) + \delta^n(t)X_t} \end{pmatrix} dW_t^Q,$$

[1] Krippner (2006) derives a special case of the AFNS model with constant risk premiums.

[2] Note that (3) refers to the risk-neutral ("Q") dynamics.

[3] Stationarity of the state variables is ensured if the real components of all eigenvalues of $K^Q(t)$ are positive; see Ahn et al. (2002). However, stationarity is not a necessary requirement for the process to be well defined.