Physica A 388 (2009) 2839–2853

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

# The distribution of first-passage times and durations in FOREX and future markets

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ARTICLE INFO

*Article history:*
Received 31 July 2008
Received in revised form 16 March 2009
Available online 26 March 2009

*PACS:*
89.65.Gh
02.50.-r

*Keywords:*
Stochastic process
Time interval distribution
Mittag–Leffler survival function
Weibull distribution
The Sony Bank USD/JPY rate
BTP futures
Average waiting time
Gini index

ABSTRACT

Possible distributions are discussed for intertrade durations and first-passage processes in financial markets. The view-point of *renewal theory* is assumed. In order to represent market data with relatively long durations, two types of distributions are used, namely a distribution derived from the Mittag–Leffler survival function and the Weibull distribution. For the Mittag–Leffler type distribution, the average waiting time (residual life time) is strongly dependent on the choice of a cut-off parameter $t_{max}$, whereas the results based on the Weibull distribution do not depend on such a cut-off. Therefore, a Weibull distribution is more convenient than a Mittag–Leffler type if one wishes to evaluate relevant statistics such as average waiting time in financial markets with long durations. On the other hand, we find that the Gini index is rather independent of the cut-off parameter. Based on the above considerations, we propose a good candidate for describing the distribution of first-passage time in a market: The Weibull distribution with a power-law tail. This distribution compensates the gap between theoretical and empirical results more efficiently than a simple Weibull distribution. It should be stressed that a Weibull distribution with a power-law tail is more flexible than the Mittag–Leffler distribution, which itself can be approximated by a Weibull distribution and a power-law. Indeed, the key point is that in the former case there is freedom of choice for the exponent of the power-law attached to the Weibull distribution, which can exceed 1 in order to reproduce decays faster than possible with a Mittag–Leffler distribution. We also give a useful formula to determine an optimal crossover point minimizing the difference between the empirical average waiting time and the one predicted from renewal theory. Moreover, we discuss the limitation of our distributions by applying our distribution to the analysis of the BTP future and calculating the average waiting time. We find that our distribution is applicable as long as durations follow a Weibull law for short times and do not have too heavy a tail.



## 1. Introduction

The distribution of time intervals between price changes gives us important pieces of information about the market [1]. In particular, the fact that intertrade durations are not exponentially distributed, rules out the possibility of using pure-jump Lévy stochastic processes (i.e. compound Poisson processes) as models for tick-by-tick data. Lévy processes have stationary and independent increments and are Markovian and all these properties are a consequence of exponentially distributed

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0378-4371/$ – see front matter 
doi:10.1016/j.physa.2009.03.027

waiting times [1]. Other models have been proposed such as non-homogeneous compound Poisson processes, GARCH-ACD models, continuous-time random walks and semi-Markov processes [2–8].

Recently, various on-line trading services on the internet were established by several major banks. For instance, the Sony Bank uses a trading system in which foreign currency exchange rates change according to a first-passage process. Namely the Sony Bank USD/JPY exchange rate is updated only when a reference market rate fluctuates by more than or equal to 0.1 yen [9]. As a result, in the case of the Sony Bank rate, the average duration between price changes becomes longer, passing from 7 s to 20 min. Automatic FOREX trading systems such as the one offered by the Sony Bank are very popular in Japan where many investors use a scheme called *carry trade* by borrowing money in a currency with low interest rate and lending it in a currency offering higher interest rates. As Japanese bond yields are low and US bonds offer higher interest rates and are rated as safe financial instruments, there is much trade in the USD/JPY market.

In this paper, we wish to compare the time structure of the Sony Bank trades with other markets such as BTP futures (BTP is the middle and long term Italian Government bonds with fixed interest rates) once traded at LIFFE (LIFFE stands for London International Financial Futures and Options Exchange).

From the viewpoint of complex system engineering, a relevant quantity used to specify the stochastic process of the market rate is the average waiting time (a.k.a. residual life time) rather than the average duration. Notice that here, the average waiting time is defined as the average time between the price observation and the arrival of the next price variation. The average duration is the average time interval between consecutive price changes. In a series of recent studies by the present authors, the average waiting time of the Sony Bank USD/JPY exchange rate was evaluated under the assumption that the first-passage time (FPT) [10–14] is a renewal process whose distribution obeys a Weibull law. We found that, counter-intuitively, the average waiting time of Sony Bank USD/JPY exchange rate is more than twice of the average duration [15]. This fact is known as *inspection paradox* [16]. It means in general that the average of durations is shorter than the average waiting time. This fact is quite counter-intuitive because the customer checks the rate at the time between arbitrary consecutive rate changes. We shall explain the interpretation of this fact for the case in which durations follow the Weibull distribution.

The Weibull distribution [17] is often used for modelling intertrade durations in financial markets [4,18]. The Weibull distribution includes the exponential distribution as a special case and the difference from the exponential distribution can be quantified using only one parameter. On the other hand, the Mittag–Leffler survival function [19–21] has been also proposed to represent the complementary cumulative distribution of durations in several markets. For example, Mainardi et al. [22] showed that BTP future inter-trade durations are well-described by a survival function of the Mittag–Leffler type. Also, for what concerns phenomenological modelling, the Mittag–Leffler function plays an important role in the solution of time-fractional diffusion equations. However, up to now, the Mittag–Leffler survival function has never been applied to the evaluation of the average waiting time as it has infinite moments of any integer order.

In this paper, we compare a Weibull distribution with a Mittag–Leffler survival function in order to evaluate the average waiting time. We give an analytical formula for the average waiting time under the assumption that the FPT distribution might be described by a Mittag–Leffler survival function. We find that the average waiting time diverges linearly with respect to a cut-off parameter $t_{\max}$. This fact tells us that it is hard to handle the Mittag–Leffler survival function to evaluate relevant statistics such as the average waiting time. We next evaluate the Gini index as another relevant statistic to check the usefulness of the Mittag–Leffler survival function.

We also provide a good candidate for the description of the first-passage process of the market rates, namely a Weibull distribution in which the behavior of the distribution changes from a Weibull law to a power-law at some crossover point $t_{\times}$. We find that the average waiting time becomes much closer to the empirical value for the Sony Bank USD/JPY exchange rate than for a pure Weibull distribution. We also give a useful formula to determine the optimal crossover point in the sense that the gap of the average waiting time between the empirical and the proposed distributions is minimized for the crossover point. Moreover, we discuss the limitation of our distribution by applying our distribution to the analysis of the BTP future and calculating the average waiting time. We find that our distribution is applicable as long as duration follows a Weibull law in short duration regime and does not have too heavy a tail.

As mentioned above, in this paper, two sets of data are used. The first set comes from the Sony Bank and the random variable analyzed is a *first-passage time*, whereas the second set is made up of future BTP prices traded at LIFFE in 1997 for two different maturities: June and September. For these data, the relevant random variable is an *intertrade duration*. Both data sets have already been studied and extensively described in previous papers ([15,22–26]). In both cases, we assume that the empirical random variables are a realization of a renewal process. A renewal process is a one-dimensional point process where at times $T_0, T_1, \ldots, T_n, \ldots$ some event takes place, and the differences $\tau_i = T_i - T_{i-1}$ are independent and identically distributed (i.i.d.) random variables, so that $T_n = \sum_{i=1}^{n} \tau_i$.: Therefore $T_n$ can be seen as a sum of non-negative i.i.d. random variables, that is as an instance of random walk. For the Sony Bank data the incoming events are price changes due to crossing the $\pm 0.1$ yen level around the current price, whereas in the BTP-future case, the events are consecutive trades. Therefore, in the Sony Bank case, the waiting time is the residual life-time to the next passage, and in the BTP-future case the waiting time is the residual life-time to the next trade.

This paper is organized as follows. In the next section, we introduce both the Mittag–Leffler survival function and the Weibull distribution. Then we discuss their properties in detail. In Section 3, we evaluate the average waiting time for the Mittag–Leffler survival function. We find that the average waiting time diverges linearly as a function of the cut-off parameter $t_{\max}$. In Section 4, we provide a theoretical formula of the Gini index for the Mittag–Leffler function and we