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# Fuzzy data envelopment analysis: A fuzzy expected value approach ☆

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ABSTRACT

Performance assessment often has to be conducted under uncertainty. This paper proposes a "fuzzy expected value approach" for data envelopment analysis (DEA) in which fuzzy inputs and fuzzy outputs are first weighted, respectively, and their expected values then used to measure the optimistic and pessimistic efficiencies of decision making units (DMUs) in fuzzy environments. The two efficiencies are finally geometrically averaged for the purposes of ranking and identifying the best performing DMU. The proposed fuzzy expected value approach and its resultant models are illustrated with three numerical examples, including the selection of a flexible manufacturing system (FMS).



## 1. Introduction

Traditional data envelopment analysis (DEA) (Charnes, Cooper, & Rhodes, 1978) requires crisp input and output data, which may not always be available in real word applications. Significant efforts have been made to handle fuzzy input and fuzzy output data in DEA. For example, Sengupta (1992) incorporated fuzziness into DEA by defining tolerance levels for both the objective function and violations of constraints and proposed a fuzzy mathematical programming approach. Triantis and Girod (1998) transformed fuzzy input and fuzzy output data into crisp data using membership function values and suggested a mathematical programming approach in which efficiency scores were computed for different values of membership functions and then averaged. Guo and Tanaka (2001) converted fuzzy constraints such as fuzzy equalities and fuzzy inequalities into crisp constraints by predefining a possibility level and using the comparison rule for fuzzy numbers and presented a fuzzy CCR model. León, Liern, Ruiz, and Sirvent (2003) suggested a fuzzy BCC model based on the same idea.

Lertworasirikul, Fang, Joines, and Nuttle (2003) proposed a possibility DEA model for fuzzy DEA. In the special case that fuzzy data are trapezoidal fuzzy numbers, the possibility DEA model became a linear programming (LP) model. They (Lertworasirikul, Fang, Joines, & Nuttle, 2003) also presented a credibility approach as an alternative way for solving fuzzy DEA problems. The possibility and credibility approaches were further extended to fuzzy BCC model in Lertworasirikul, Fang, Nuttle, and Joines (2003) by the same authors. Wu, Yang, and Liang (2006) applied the possibility DEA model for efficiency analysis of cross-region bank branches in Canada. Garcia, Schirru, and Melo (2005) utilized the possibility DEA model for failure mode and effects analysis (FMEA) and presented a fuzzy DEA approach to determining ranking indices among failure modes. Wen and Li (2009) employed credibility measure to represent fuzzy CCR model as an uncertain programming and solved it with a hybrid intelligent algorithm which integrates fuzzy simulations and genetic algorithms.

Kao and Liu (2000a, 2000b, 2003, 2005) transformed fuzzy input and fuzzy output data into intervals by using $\alpha$-level sets and Zadeh's extension principle, and built a family of crisp DEA models for the intervals. Based on their crisp DEA models for $\alpha$-level sets, Liu (2008) and Liu and Chuang (2009) took further into consideration the concept of assurance region (AR) and developed a fuzzy DEA/AR model for the selection of flexible manufacturing systems (FMSs) and the assessment of university libraries, respectively. Saati, Menariani, and Jahanshahloo (2002) defined fuzzy CCR model as a possibilistic-programming problem and transformed it into an interval programming by specifying a $\alpha$-level set. Their approach was further extended in Saati and Memariani (2005) so that all decision making units (DMUs) could be evaluated with a common set of weights under a given $\alpha$-level set. Entani, Maeda, and Tanaka (2002) and Wang, Greatbanks, and Yang (2005) also changed fuzzy input and fuzzy output data into intervals by using $\alpha$-level sets, but suggested two different interval DEA models.

Dia (2004) proposed a fuzzy DEA model based upon fuzzy arithmetic operations and fuzzy comparisons between fuzzy numbers. The model requires the decision maker (DM) to specify a fuzzy aspiration level and a safety $\alpha$-level so that the fuzzy DEA model could be transformed into a crisp DEA model for solution. Wang, Luo, and Liang (2009) constructed two fuzzy DEA models from the perspective of fuzzy arithmetic to deal with fuzziness in input

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and output data in DEA. The two fuzzy DEA models were both formulated as linear programs and could be solved to determine fuzzy efficiencies of DMUs.

Triantis (2003) introduced a fuzzy DEA approach to calculate fuzzy non-radial technical efficiencies and implemented the approach in a newspaper preprint insertion manufacturing process. Soleimani-damaneh, Jahanshahloo, and Abbasbandy (2006) addressed some computational and theoretical pitfalls of the fuzzy DEA models developed in Kao and Liu (2000a), León et al. (2003) and Lertworasirikul et al. (2003) and provided a fuzzy DEA model to yield crisp efficiencies for the DMUs with fuzzy input and fuzzy output data. Jahanshahloo, Soleimani-damaneh, and Nasrabadi (2004) extended a slack-based measure (SBM) of efficiency in DEA to fuzzy settings and developed a two-objective nonlinear DEA model for fuzzy DEA.

Existing fuzzy DEA models exhibit some drawbacks. For instance, fuzzy DEA models derived from the direct fuzzification of crisp DEA models ignore the fact that a fuzzy fractional program cannot be transformed into an LP model in the traditional way that we do for a crisp fractional program. Fuzzy DEA models built on the basis of $\alpha$-level sets require the solution of a series of LP models and thus considerable computational efforts. Fuzzy DEA models constructed from the perspective of fuzzy arithmetic demand a rational yet easy-to-use ranking approach for fuzzy efficiencies. To overcome these drawbacks, we propose in this paper a "fuzzy expected value approach" for fuzzy DEA, which first weights fuzzy inputs and fuzzy outputs, respectively, and then utilizes their expected values for measuring the performances of DMUs in fuzzy environments.

The paper is organized as follows. Section 2 introduces the measures of fuzzy expected values and develops fuzzy expected value models for fuzzy DEA. Section 3 illustrates the developed fuzzy expected value models with three numerical examples, including the selection of a FMS. Section 4 concludes the paper.

## 2. Fuzzy expected values and fuzzy DEA models

Fuzzy numbers are convex fuzzy sets, characterized by given intervals of real numbers, each interval with a grade of membership between 0 and 1. The most commonly used fuzzy numbers are triangular and trapezoidal fuzzy numbers defined by the following membership functions, respectively:

$$\mu_{\widetilde{A}_1}(x)=\begin{cases}(x-a)/(b-a), & a\leqslant x\leqslant b,\\ (d-x)/(d-b), & b\leqslant x\leqslant d,\\ 0, & \text{otherwise},\end{cases}\tag{1}$$

$$\mu_{\widetilde{A}_2}(x)=\begin{cases}(x-a)/(b-a), & a\leqslant x\leqslant b,\\ 1, & b\leqslant x\leqslant c,\\ (d-x)/(d-c), & c\leqslant x\leqslant d,\\ 0, & \text{otherwise}.\end{cases}\tag{2}$$

For brevity, triangular and trapezoidal fuzzy numbers are often denoted as $(a,b,d)$ and $(a,b,c,d)$. It is evident that triangular fuzzy numbers are special cases of trapezoidal fuzzy numbers with $b = c$. For any two positive trapezoidal fuzzy numbers $\widetilde{A}=(a_L,a_M,a_N,a_U)$ and $\widetilde{B}=(b_L,b_M,b_N,b_U)$, fuzzy addition and fuzzy multiplication on $\widetilde{A}$ and $\widetilde{B}$ are respectively defined as $\widetilde{A}+\widetilde{B}=(a_L+b_L,a_M+b_M,a_N+b_N,a_U+b_U)$ and $\widetilde{A}\times\widetilde{B}\approx(a_Lb_L, a_Mb_M,a_Nb_N,a_Ub_U)$.

Let $\xi$ be a fuzzy variable with a membership function $\mu$: $\Re\to[0,1]$ and $r$ be a real number. The possibility and the necessity of $\{\xi\geqslant r\}$ are respectively defined by

$$Pos\{\xi\geqslant r\}=\sup_{x\geqslant r}\mu(x),\tag{3}$$

$$Nec\{\xi\geqslant r\}=1-Pos\{\xi<r\}=1-\sup_{x<r}\mu(x),\tag{4}$$

which show respectively the possibility and the necessity degrees to which $\xi$ is not smaller than $r$. $Pos$ and $Nec$ are a pair of dual fuzzy measures in the sense that $Pos\{A\} = 1 - Nec\{A^C\}$ with $A^C$ is the complement of A. Based upon the possibility and the necessity measures, credibility measure is defined as

$$Cr\{\xi\geqslant r\}=\frac{1}{2}(Pos\{\xi\geqslant r\}+Nec\{\xi\geqslant r\}).\tag{5}$$

The fuzzy expected value of $\xi$ can thus be defined as (Liu & Liu, 2002)

$$E[\xi]=\int_0^{\infty}Cr\{\xi\geqslant r\}dr-\int_{-\infty}^{0}Cr\{\xi\leqslant r\}dr.\tag{6}$$

It has been shown (Liu & Liu, 2002) that if fuzzy variable $\xi$ is replaced with a random variable whose probability density function is $\phi$ and $Cr$ is replaced with the probability measure Pr, then there exists $\int_0^{\infty}\Pr\{\xi\geqslant r\}dr-\int_{-\infty}^{0}\Pr\{\xi\leqslant r\}dr=\int_{-\infty}^{+\infty}x\phi(x)dx$, which is exactly the expected value of the random variable $\xi$.

It is also shown (Liu & Liu, 2002) that if $\xi$ is a trapezoidal fuzzy variable $(r_1,r_2,r_3,r_4)$, then the expected value of $\xi$ is $(1/4)(r_1+r_2+r_3+r_4)$. In particular, if $\xi$ is a triangular fuzzy variable $(r_1,r_2,r_3)$, then the expected value of $\xi$ is $(1/4)(r_1+2r_2+r_3)$.

Suppose we have $n$ DMUs to be evaluated in terms of $m$ inputs and $s$ outputs. Let $x_{ij}$ $(i = 1,\ldots,m)$ and $y_{rj}$ $(r = 1,\ldots,s)$ be the input and output data of DMU$_j$ $(j = 1,\ldots,n)$. Without loss of generality, all input and output data $x_{ij}$ and $y_{rj}$ are assumed to be uncertain and characterized by trapezoidal fuzzy numbers $\tilde{x}_{ij}=\left(x_{ij}^L,x_{ij}^M,x_{ij}^N,x_{ij}^U\right)$ and $\tilde{y}_{rj}=\left(y_{rj}^L,y_{rj}^M,y_{rj}^N,y_{rj}^U\right)$ with $x_{ij}^L\geqslant 0$ and $y_{rj}^L\geqslant 0$ for $i = 1$ to $m$, $r = 1$ to $s$, and $j = 1$ to $n$. Crisp data and triangular fuzzy data are treated as special cases of trapezoidal fuzzy data $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ with $x_{ij}^L=x_{ij}^M=x_{ij}^N=x_{ij}^U$, $y_{rj}^L=y_{rj}^M=y_{rj}^N=y_{rj}^U$, and $x_{ij}^M=x_{ij}^N$, $y_{rj}^M=y_{rj}^N$, respectively. The total fuzzy weighted output (FWO) and the total fuzzy weighted input (FWI) of DMU$_j$ are given by

$$FWO_j=\sum_{r=1}^{s}\tilde{u}_r\tilde{y}_{rj}=\sum_{r=1}^{s}\left(u_r^L,u_r^M,u_r^N,u_r^U\right)\times\left(y_{rj}^L,y_{rj}^M,y_{rj}^N,y_{rj}^U\right),\tag{7}$$

$$FWI_j=\sum_{i=1}^{m}\tilde{v}_i\tilde{x}_{ij}=\sum_{i=1}^{m}\left(v_i^L,v_i^M,v_i^N,v_i^U\right)\times\left(x_{ij}^L,x_{ij}^M,x_{ij}^N,x_{ij}^U\right),\tag{8}$$

where $\tilde{u}_r=(u_r^L,u_r^M,u_r^N,u_r^U)$ and $\tilde{v}_i=(v_i^L,v_i^M,v_i^N,v_i^U)$ are fuzzy weights for fuzzy input $\tilde{x}_{ij}$ and fuzzy output $\tilde{y}_{rj}$, respectively. According to fuzzy addition and fuzzy multiplication operations on two positive fuzzy numbers, (7) and (8) can be approximately expressed as

$$FWO_j\approx\left(\sum_{r=1}^{s}u_r^Ly_{rj}^L,\sum_{r=1}^{s}u_r^My_{rj}^M,\sum_{r=1}^{s}u_r^Ny_{rj}^N,\sum_{r=1}^{s}u_r^Uy_{rj}^U\right),\tag{9}$$

$$FWI_j\approx\left(\sum_{i=1}^{m}v_i^Lx_{ij}^L,\sum_{i=1}^{m}v_i^Mx_{ij}^M,\sum_{i=1}^{m}v_i^Nx_{ij}^N,\sum_{i=1}^{m}v_i^Ux_{ij}^U\right),\tag{10}$$

which can be viewed as two trapezoidal fuzzy variables, whose expected values can therefore be determined as

$$\begin{aligned}E(FWO_j)&=\frac{1}{4}\left(\sum_{r=1}^{s}u_r^Ly_{rj}^L+\sum_{r=1}^{s}u_r^My_{rj}^M+\sum_{r=1}^{s}u_r^Ny_{rj}^N+\sum_{r=1}^{s}u_r^Uy_{rj}^U\right)\\&=\frac{1}{4}\sum_{r=1}^{s}\left(u_r^Ly_{rj}^L+u_r^My_{rj}^M+u_r^Ny_{rj}^N+u_r^Uy_{rj}^U\right),\end{aligned}\tag{11}$$