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Counting combinatorial choice rules

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Abstract

I count the number of combinatorial choice rules that satisfy certain properties: *Kelso–Crawford substitutability*, and *independence of irrelevant alternatives*. The results are important for two-sided matching theory, where agents are modeled by combinatorial choice rules with these properties. The rules are a small, and asymptotically vanishing, fraction of all choice rules. But they are still exponentially more than the preference relations over individual agents—which has positive implications for the Gale–Shapley algorithm of matching theory.

JEL classification: C65; C78

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1. Introduction

Consider hiring a team of workers from a set A of available workers. The decision of hiring worker x is not independent of the decision to hire worker y; the workers may be complements or substitutes. Let $C(A) \subseteq A$ be the workers hired. The function C is called a (combinatorial) choice rule. I shall give results on the number of functions C that satisfy various properties.

The main application I have in mind is the theory of matching markets (Roth and Sotomayor, 1990). In many-to-one, and many-to-many, matching theory, some agents, "firms," are matched to a set of individual "workers." Firms' behavior is modeled as a combinatorial choice rule: for each set of available workers they select a subset.

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The classical results on matching markets—among others, that the core is nonempty—require structure on the firms' choice rules: *substitutability* and *independence of irrelevant alternatives* (IIA). Substitutability was introduced, and applied to matching markets, by Kelso and Crawford (1982). It requires that, if worker x is chosen out of a set that contains worker y, then x is also chosen when y is not available. So x could not have been chosen because she was complementary to y, hence the name substitutability (see Section 2.1 for a formal definition and comparison with the assumption in Kelso and Crawford). IIA is a more basic rationality assumption. Substitutability and IIA are sufficient for the classical results on matching markets (this is clear from Roth, 1984 and Blair, 1988).

Substitutability and IIA have played a central role in the research on matching markets since Kelso and Crawford (1982), and on related problems, e.g. Gul and Stacchetti (1999, 2000) and Hatfield and Milgrom (2005) use substitutability in models that extend beyond matching models. But, while there are several analysis of substitutable combinatorial choice rules—Gul and Stacchetti (1999), Beviá et al. (1999), and Fujishige and Yang (2003) are recent examples—nobody has counted them.

I count the number of choice rules that satisfy substitutability and IIA. The results, and their main implications, are:

(1) The choice rules that satisfy substitutability are a small, and asymptotically vanishing, fraction of all choice rules. Arguably, then, substitutability is a strong assumption. In continuous models, one routinely disregards cases with Lebesgue-measure zero. The same logic suggests that substitutability is a strong assumption, in the sense that it is almost a "knife-edge" case. Concretely, for large groups of workers, the structure that ensures a stable matching is unlikely. I should make two qualifications:
First, an obvious caveat is: Even if they are scarce, the substitutable choice rules may nevertheless often occur. For example, because they are induced by certain behaviors—such as "responsive" preferences (Roth and Sotomayor, 1990, p. 173).
Second, one may also want to compare the number of substitutable rules to more restrictive sets of rules. With some knowledge about agents' behavior in the problem at hand, one can take some properties as given. It may then be that substitutability is not a strong additional assumption, if the substitutable rules are not scarce relative to the smaller set of rules.

(2) The choice rules that satisfy substitutability and IIA are exponentially more than the preference relations over individual workers. So the choice rules with the structure used in matching theory are few, *ma non troppo*.
This result has an important implication for the Gale–Shapley algorithm for finding a matching in the core of the many-to-one matching market. Segal (2003) proves that the Gale–Shapley algorithm requires approximately as much communication as communicating a preference relation over individual workers. My results and Segal's then imply that the algorithm requires exponentially less communication than full revelation of agents' choice rules. The implication helps explain why the Gale–Shapley algorithm is so widely used in practice. See Segal (2003) on what communication means, and why full revelation is the right benchmark.

The numbers involved are surprising. Suppose 8 objects can be chosen, much fewer than in actual matching markets. Already with 8 objects, the substitutable choice rules are a small fraction of the number of choice rules. There are 1.8×10^{308}, roughly a centillion, different