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Option hedging theory under transaction costs

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ABSTRACT

The problem of option hedging in the presence of proportional transaction costs can be formulated as a singular stochastic control problem. Hodges and Neuberger [1989. Optimal replication of contingent claims under transactions costs. Review of Futures Markets 8, 222–239] introduced an approach that is based on maximization of the expected utility of terminal wealth. We develop a new algorithm to solve the corresponding singular stochastic control problem and introduce a new approach to option hedging which is closer in spirit to the pathwise replication of Black and Scholes [1973. The pricing of options and corporate liabilities. Journal of Political Economy 81, 637–654]. This new approach is based on minimization of a Black–Scholes-type measure of pathwise risk, defined in terms of a market delta, subject to an upper bound on the hedging cost. We provide an efficient backward induction algorithm for the problem of cost-constrained risk minimization, whose associated singular stochastic control problem is shown to be equivalent to an optimal stopping problem. This algorithm is then modified to solve the singular stochastic control problem associated with utility maximization, which cannot be reduced to an optimal stopping problem. We propose to choose an optimal parameter (risk-aversion coefficient or Lagrange multiplier) in either approach by minimizing the mean squared hedging error and demonstrate that with this "best" choice of the parameter, both approaches have similar performance. We also discuss the different notions of risk in both approaches and propose a volatility adjustment for the risk-minimization approach, which is analogous to that introduced by Zakamouline [2006. European option pricing and hedging with both fixed and proportional transaction costs. Journal of Economic Dynamics and Control 30, 1–25] for the utility maximization approach, thereby providing a unified treatment of both approaches.

1. Introduction

The problem of option pricing and hedging was initially studied in an idealized setting where an investor incurs no transaction costs from trading in a market consisting of a risk-free asset ("bond") with constant rate of return and a risky asset ("stock") whose price is a geometric Brownian motion with constant rate of return and volatility. For this setting, Black and Scholes (1973) demonstrated that in the absence of arbitrage the value of an option is an expectation of the

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discounted payoff at expiration under the "risk-neutral" measure, for which the stock's rate of return equals the risk-free rate. Moreover, perfect replication of the option is possible and the option is itself "redundant" in such a "complete" market. However, the Black–Scholes "delta-hedging" portfolio requires continuous trading. In the presence of transaction costs proportional to the amount of trading, such a continuous strategy is prohibitively expensive. Hence it is impossible to perfectly replicate the option in this setting when there are transaction costs and, as a result, trading in an option involves an essential element of risk.

One approach to characterize this hedging risk examines the difference between the realized cash flow from a hedging strategy and the desired payoff at maturity. It embeds option hedging within the framework of portfolio selection introduced by Magill and Constantinides (1976) and Davis and Norman (1990), and uses a risk-averse utility function to assess this shortfall ("replication error"). In this way, Hodges and Neuberger (1989) formulated the problem of option hedging as that of maximizing the investor's expected utility of terminal wealth. Making use of an indifference argument, the reservation selling or buying price of an option is defined as the amount of money that would make an investor indifferent, in terms of expected utility, between trading in the market with and without a (short or long) position in the option. This involves the value functions of two singular stochastic control problems and the optimal hedge is given by the difference in the trading strategies corresponding to these two problems. The nature of the optimal hedge is that an investor with an option position should rebalance his portfolio only when the number of shares of stock falls "too far" out of line. For the negative exponential utility function, Davis et al. (1993), Clewlow and Hodges (1997) and Zakamouline (2006) have developed numerical methods to compute the optimal hedge and option price by making use of discrete-time dynamic programming for an approximating binomial tree for the stock price. Whalley and Wilmott (1997) and Barles and Soner (1998) have developed asymptotic approximations for these hedging strategies and option prices as the transaction costs approach 0. Constantinides and Zariphopoulou (1999, 2001) have provided option price bounds under general utility functions (rather than the negative exponential utility function commonly adopted for numerical studies). In this paper we make use of a new numerical method for solving singular stochastic control problems, recently introduced by Lai et al. (2009), to develop a much simpler algorithm to compute the buy–sell boundaries and value functions in the utility-based approach.

In the presence of transaction costs, alternatives to the utility-based approach have been based on super-replication (or replication) in a discrete-time setting and are concerned with finding trading strategies which produce payoffs at expiration that are at least (or exactly) as valuable as the option payoff. Noting that using the Black–Scholes delta to short-sell delta shares of stock at the beginning of each revision interval introduces too high transaction costs as the width of the revision interval shrinks to 0, Leland (1985) proposed a modification of the variance used in the Black–Scholes delta so as to yield the desired option payoff at expiration inclusive of transaction costs. The fact that this modified strategy is not self-financing has prompted Boyle and Vorst (1992) to work in a discrete-state (binomial tree) framework to construct a *self-financing* discrete-time replicating strategy, thereby extending the two-period model of Merton (1990, Chapter 14). Explicit portfolio weights at each node of the binomial tree can be computed by using a backward induction procedure. However, these methods require the user to exogenously specify a revision interval and it is unclear how one can do so optimally. In fact, as the width of the revision interval approaches 0, the cost of the option approaches the price of a single share of stock, which turns out to be the least expensive way of super-replicating the option in a continuous-time model; see Soner et al. (1995). For the binomial tree model, Bensaid et al. (1992) derived bounds on the option value at inception by minimizing the initial cost of the self-financing strategy used to produce a super-replicating portfolio of stock and bond at expiration. As they have shown, by rebalancing only in the earlier periods, it is possible to have a super-replicating portfolio that is *less expensive* than the corresponding replicating portfolio. In general, the optimal discrete-time super-replicating strategy is such that the investor with an option position does not transact at a trading date if the inherited amount of stock is in a certain range (which depends on the past history of the stock price); otherwise he adjusts his portfolio back to this range. Noting that this cost minimization problem associated with super-replication is path dependent and that the dynamic programming algorithm is computationally expensive if the number of periods is not sufficiently small, Edirisinghe et al. (1993) developed a linear programming algorithm and a two-stage dynamic programming method to approximate the optimal solution. More recently, Primbs (2009) provided an alternative formulation of super-replication in terms of the first two moments of the replication error.

In this paper we propose a new approach which formulates the option hedging problem in the presence of transaction costs as a constrained risk minimization problem that minimizes a measure of pathwise risk under the constraint that the hedging cost (central to the replication/super-replication approach) does not exceed a prescribed level. This measure of pathwise risk is implicit in the Black–Scholes theory that continuously rebalances the portfolio to make such risk zero when there are no transaction costs, and leads in our approach to a natural modification of the Black–Scholes delta-hedging scheme. In the presence of transaction costs, this modification consists of buying or selling the underlying stock whenever the holding of shares falls outside a no-transaction band containing the option's delta. The corresponding singular stochastic control problem, whose no-action region is the no-transaction band, is equivalent to an optimal stopping problem. This equivalence is used to compute the buy- and sell-boundaries efficiently, thereby reducing substantially the computational complexity of the original singular stochastic control problem, which requires determination of both *when* to apply the control (in the form of buying or selling) and *how much* control to apply.

This paper is organized as follows. Section 2 defines the hedging cost of a self-financing strategy and uses it to formulate the singular stochastic control problems associated with the utility maximization and the cost-constrained risk