ARTICLE IN PRESS
Journal of Mathematical Economics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
Contents lists available at ScienceDirect
Journal of Mathematical Economics
journal homepage: www.elsevier.com/locate/jmateco

Subtle price discrimination and surplus extraction under uncertainty

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ARTICLE INFO

Article history:
Received 24 May 2013
Accepted 7 August 2013
Available online xxxx

Keywords:
Kelly criterion
Risk management
Asset allocation
Betting rules
Price discrimination
Expected utility theory

ABSTRACT

This paper provides a solution to Proebsting's Paradox, an argument that appears to show that the investment rule known as the Kelly criterion can lead a decision maker to invest a higher fraction of his wealth the more unfavorable the odds he faces are and, as a consequence, risk an arbitrarily high proportion of his wealth on the outcome of a single event. The paper shows that a large class of investment criteria, including 'fractional Kelly', also suffer from the same shortcoming and adapts ideas from the literature on price discrimination and surplus extraction to explain why this is so. The paper also presents a new criterion, dubbed the *doubly conservative* criterion, that is immune to the problem identified above. Immunity stems from the investor's attitudes toward capital preservation and from him becoming rapidly pessimistic about his chances of winning the better odds he is offered.

1. Introduction

Proebsting's Paradox is an argument that appears to show that the investment rule known as the *Kelly criterion* can lead a decision maker to invest a higher fraction of his wealth the more unfavorable the odds he faces are and, consequently, risk an arbitrarily high proportion of his wealth on the outcome of a single event. According to this criterion one ought to choose the size of one's investment so as to maximize the expected growth of one's wealth. Addressing the paradox is important in that it seems to contradict the well known fact that a bettor that follows the Kelly criterion can never be ruined absolutely (capital equal zero) or asymptotically (capital tends to zero with positive probability).[1] The paradox was first communicated by Todd Proebsting, a computer scientist, to Ed Thorp, a mathematician, by email, who in turn made it publicly known in an article in the September 2008 issue of Wilmott Magazine, a magazine that serves the quantitative finance community.

A second reason why addressing the paradox is important is because in recent years the Kelly criterion has developed a reputation of being a part of many successful investment strategies[2] and the claim has been made that the world's most prominent stock investor, Warren Buffett, and the world's most prominent bond investor, Bill Gross, both allocate capital in manners that are consistent with the Kelly criterion.[3] Ed Thorp himself, perhaps the main proponent of the Kelly criterion in the gambling and investment community (and a very successful hedge fund manager in his own right)[4] explicitly employs the Kelly criterion as his chief portfolio allocation rule. It has even been argued that the risk–return characteristics of the very successful investments John Maynard Keynes made on behalf of King's College Cambridge's Chest Fund from 1927 to 1945 are very similar to those generated by the Kelly criterion (Ziemba, 2005).

A third reason why addressing the paradox is important is because of the long standing (and somewhat unresolved) feud that has existed since the early 1970s between the proponents of the Kelly criterion (such as Ed Thorp and, in his time, Claude Shannon) and its opponents (prominently, Nobel Laureates Paul Samuelson and Robert Merton—among others). The opponents argue that it is irrelevant that the Kelly criterion maximizes the expected rate of growth of the individual's wealth if this criterion leads to a portfolio with risk levels that the individual is not willing to tolerate. They say that to properly advise an individual regarding portfolio choice one has to identify the individual's attitudes toward risk and construct a portfolio based on those, without regard to what the Kelly criterion would prescribe. Proponents of the Kelly criterion would respond by saying things like: "Of course this is the case, but it does

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1 See, e.g., Thorp (1969, 2008) and Breiman (1961). For a comprehensive survey on the Kelly criterion and its applications see Thorp (2006).

2 See, e.g., MacLean et al. (2010). A lay reader account of the Kelly criterion's influence on the academic and investment communities can be found in Poundstone (2005).

3 See, e.g., Thorp (2006) and Gross (2006).

4 Thorp used to run two hedge funds, Princeton–Newport Partners and Ridgeline Partners, which went nearly 30 years without a down year, and averaged 19%–20% annual returns. See Patterson (2008).

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http://dx.doi.org/10.1016/j.jmateco.2013.08.004

not deny the fact that (the Kelly criterion) has an objective property: it has a better growth rate than that achieved by any other strategy" (Cover, 2008). As a response to arguments of this type Samuelson (1979) wrote a two page paper in 1979 which, in words of one syllable, argued that maximizing the expected growth rate of wealth need not be appropriate. Later on, Samuelson went on to call the Kelly criterion a "complete swindle" (Poundstone, 2005). The debate stalled more or less back then.

This paper offers some bad news and some "good news" for the proponents of the Kelly criterion. The bad news is that the problem is worse than what Proebsting and Thorp anticipated in that the following 'skimming' result can be proved: one can devise a sequence of *structured* investments, or bets, that a Kelly bettor would willingly accept that would entice the investor to risk virtually his entire wealth and that will keep the expected growth of his wealth as close to zero as one wished. The paper also shows that the so-called fractional Kelly criterion (a criterion that follows from betting only a constant fraction of what the Kelly criterion would dictate) is also vulnerable in the same way.

The "good news" is that the findings of Proebsting and Thorp, and the generalizations discussed above, cannot be used to argue against the use of the Kelly criterion because many other investment criteria, precisely all those that Samuelson and Merton would advocate,[5] also suffer from exactly the same shortcomings. Something deeper is generating these vulnerabilities, and the paper discusses what that is: the simple fact that, embedded in most reasonable betting systems one can devise lies the maxim "Good odds are worth paying for". Once this is in place, a version of the surplus-extracting *skimming* result described above basically goes through.[6]

There are exceptions, however: investors that would *very rapidly* become pessimistic about their chances of winning the better odds they are offered. The paper shows the existence of a family of investment rules with these characteristics (all 'distant cousins' of fractional Kelly). This family is somewhat immune to the *skimming* result in that there is a limit to how much those bettors will bet no matter how attractive the odds in any sequence of bets presented to them may be. This provides a half open door[7] out of the dismal world Proebsting and Thorp discovered exists for bettors who like betting, and from which they would find hard to escape. On that less pessimistic note, the paper ends.

The rest of the paper is devoted to a coherent presentation of the claims made above, together with additional commentary that would aid in the interpretation of what lies beneath Proebsting's paradox and its variants. The concluding section briefly discusses the implications of the results presented to the issues related to the structuring of financial securities, mutual fund design, and "plain vanilla" betting.

For expositional simplicity I frame all decision problems below using the terminology of *fixed-odds betting*. Bettors will be regarded as male while bookies will be regarded as female.

2. The paradox[8]

Suppose that you believe an event will occur with 50% probability and somebody offers you 2–1 odds on that event. How much money you should place on this bet depends on how you feel about the tradeoff between risk and reward that is being offered to you. If you were *neutral to risk* and all you cared about was the expected final value of wealth, then you would place 100% of your wealth on the gamble, and you would ignore the fact that you may end up losing everything with probability 50%. The Kelly criterion would have you be much more conservative than that in that it would instead have you focus on the expected growth rate of your wealth. In the case in which you are offered the 2–1 odds (call this "Situation G", for 'good'), the task is to find the fraction f^G of your wealth that solves the problem

$$\max_f \frac{1}{2}\ln(W - fW) + \frac{1}{2}\ln(W + 2fW),$$

which yields $f^G = 0.25$ and exposes you to a far lower risk[9] than if you place all your money on the bet. More generally, if you are offered a 50/50 bet that pays b to 1 the Kelly criterion would have you bet a fraction of your wealth equal to $f^* = \frac{b-1}{2b}$. Hence, if you were offered 5–1 odds ("Situation B", for 'better') according to the Kelly criterion you would place $f^B = 40\%$ on your wealth on the bet.

Now, suppose that these bets occur in sequence. You are offered 2–1 odds, bet 25% of your wealth and then are offered 5–1 odds ("Situation M", for 'mixed'). Should you place an additional bet and, if so, how much?

The Kelly criterion will indeed have you place an additional bet, which can be computed as follows:

$$\max_f \frac{1}{2}\ln(W - 0.25W - fW) + \frac{1}{2}\ln(W + 2 \times 0.25W + 5fW)$$

which yields $f^* = 0.225$.

The paradox is that the total bet in this situation, $f^M = 0.25 + 0.225 = 0.475$, is larger than the 0.4 Kelly fraction if the 5–1 odds are offered from the beginning. It is counterintuitive that you bet more when some of the bet is at unfavorable odds.[10] Todd Proebsting emailed Ed Thorp asking about this.[11]

Moreover, Thorp showed that if a gambler is offered 2–1 odds, then 4–1, then 8–1, and so on, the Kelly criterion would have you eventually bet your entire wealth, thus exposing the bettor to a risk of complete ruin of exactly 50%, just as if he was risk neutral. This appears to challenge the view commonly held of the Kelly criterion keeping the investor away from any risk of ruin.

3. The resolution

3.1. *"The bettor bets more at blended 2–1 and 5–1 odds than at 5–1 odds"*

While it is correct that the bettor is facing blended (average) 2–1 and 5–1 in Situation M, what matters, for the purpose of decision making is not the *average* odds but the *marginal* odds. The odds that a bettor faces determine the rate at which the individual can sacrifice money-in-the-event-of-losing to money-in-the-event-of-winning. In particular, when the bettor is offered 5–1 odds he can sacrifice one dollar when losing in exchange for five dollars when winning. The fact that the bettor already made bets at 2–1 odds does not alter the terms of the current 5–1 tradeoff. What

5 That is, all those based on *expected utility maximization*.

6 Skimming as used in the paper is a natural variant on the practices known as "price discrimination" in the economics literature. See Varian (1989).

7 'Half open' because the investors can still be skimmed, just not completely.

8 This section is based on the account of Proebsting's Paradox given in Thorp (2008) and Wikipedia (2009).

9 And a lower expected final value of your wealth.

10 Although quite obvious, it will be important in what follows to stress that f^M is the fraction of the individual's original wealth that the Kelly bettor will ultimately bet in Situation M.

11 In general, if a bettor makes the Kelly bet on a 50/50 bet with a payout of b_G, and then is offered $b_B > b_G$, the bettor will, in this situation, bet a total of

$$f^M = f^B + f^G \frac{(b_B - b_G)}{2b_B},$$

where f^i is the Kelly bet for Situation i. From this one can tell that $f^M > f^B$.