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Scaling and long-range dependence in option pricing I: Pricing European option with transaction costs under the fractional Black–Scholes model☆

Xiao-Tian Wang

Department of Mathematics, South China University of Technology, Guangzhou, 510640, Guangdong, China

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ABSTRACT

This paper deals with the problem of discrete time option pricing by the fractional Black–Scholes model with transaction costs. By a mean self-financing delta-hedging argument in a discrete time setting, a European call option pricing formula is obtained. The minimal price $C_{\min}(t, S_t)$ of an option under transaction costs is obtained as timestep $\delta t = \left(\frac{2}{\pi}\right)^{\frac{1}{2H}} \left(\frac{k}{\sigma}\right)^{\frac{1}{H}}$, which can be used as the actual price of an option. In fact, $C_{\min}(t, S_t)$ is an adjustment to the volatility in the Black–Scholes formula by using the modified volatility $\sigma\sqrt{2}\left(\frac{2}{\pi}\right)^{\frac{1}{2}-\frac{1}{4H}} \left(\frac{k}{\sigma}\right)^{1-\frac{1}{2H}}$ to replace the volatility σ, where $\frac{k}{\sigma} < \left(\frac{\pi}{2}\right)^{\frac{1}{2}}$, $H > \frac{1}{2}$ is the Hurst exponent, and k is a proportional transaction cost parameter. In addition, we also show that timestep and long-range dependence have a significant impact on option pricing.

1. Introduction

Over the last few years, the financial markets are regarded as complex and nonlinear dynamic systems. A series of studies have found that many financial market time series display long-range dependence and momentum [1–4], which imply that there exists arbitrage in financial markets.

In classical finance theory, absence of arbitrage is one of the most unifying concepts. However, behavioral finance and econophysics as well as empirical studies sometime propose models for asset price that are not consistent with this basic assumption. A case is the fractional Black–Scholes model, which displays the long-range dependence observed in empirical data [4–13]. The fractional Black–Scholes model is a generalization of the Black–Scholes model, which is based on replacing the standard Brownian motion by a fractional Brownian motion in the Black–Scholes model. Since fractional Brownian motion (fBm) is not a semimartingale [14], it has been shown that the fractional Black–Scholes model admits arbitrage in a complete and frictionless market [15–19]. The objective of the present paper is to resolve this contradiction between classical Black–Scholes–Merton theory and practice through both giving up the arbitrage argument used by Black and Scholes to price options and examining option replication in the presence of proportional transaction costs in a discrete time setting. Transaction costs lead to the failure of the no-arbitrage principle and the continuous time trade in general: instead of no arbitrage, the principle of hedge pricing – according to which the price of an option is defined as the minimum level of initial wealth needed to hedge the option – comes to the fore. In the past decade there has been a renewed interest in analyzing the transaction cost effect on security prices and on agents' behavior in financial markets. In a complete financial market without transaction costs, the Black–Scholes no-arbitrage argument provides a hedging portfolio that replicates the option. However,

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E-mail address: swa001@126.com.

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the Black–Scholes hedging portfolio requires continuous trading and therefore, in a market with propositional transaction costs, it is expensive. In fact, in such a market, as Leland mentioned in 1985 [20], there is no portfolio that replicates the European call option and we are forced to relax the hedging condition, requiring the portfolio only to replicate the value of option at discrete times. Furthermore, as Leland pointed out, the arbitrage argument used by Black and Scholes to price options no longer can be used: because replicating the option by dynamic strategy would be infinitely costly, no effective option bounds are implied [20]. On the other hand, from the point of view of behavioral finance it is also reasonable that no-arbitrage argument is discarded [21,22]. Finally, as mentioned in the paper [23], in most models of stock fluctuations, except for very special cases, risk in option trading cannot be eliminated and strict arbitrage opportunities do not exist, whatever be the price of the option. The risk cannot be eliminated is furthermore the fundamental reason for the very existence of option markets.

The paper's outline is as follows: in Section 2 the option pricing problem on the fractional Black–Scholes model with transaction costs is considered and the scaling behavior of option pricing is analyzed. A specific example of the European call option is studied and a closed form representation of the option pricing formula is given. In Section 3, a conclusion is given.

2. An option pricing model for a fractional economy under transaction costs

Let $\left(\Omega, F, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$ be a complete probability space carrying a fractional Brownian motion $(B_H(t))_{t\in R}$ with Hurst exponent $H \in (0, 1)$, i.e. a continuous, centered Gaussian process with covariance function [6]

$$cov\,(B_H(t)B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad s, t \in R;$$

where we assume that $\mathcal{F} = \mathcal{F}_T$ for some $T \in (0, +\infty)$, the sub-σ-algebras $\mathcal{F}_t = \beta\left(B_H(\tau), \tau \in [0, t]\right)$ satisfy the usual assumptions of right continuity and saturatedness, $\mathcal{F}_0$ is trivial, and P is the real world probability measure. More details about the fBm and P can be found in the paper [24, and references therein].

If $H = \frac{1}{2}$, then the corresponding fractional Brownian motion is the usual standard Brownian motion. It can be easily seen that $E\,(B_H(t) - B_H(s))^2 = |t-s|^{2H}$. Furthermore, $B_H(t)$ has stationary increments and is H-self-similar, that is, for all $a > 0$, $B_H(at)_{t\in R}$ has the same distribution as $\left(a^H B_H(t)\right)_{t\in R}$.

If $H > \frac{1}{2}$, the process $(B_H(t),\ t \geq 0)$ exhibits a long-range dependence, that is, if $r\,(n) = E\left[B_H(1)\,(B_H(n+1) - B_H(n))\right]$, then $\sum_{n=1}^{\infty} r\,(n) = \infty$. As mentioned in [3], long-range dependence is widespread in economics and finance and has remained a topic of active research (e.g., see [6] for details, and [11–13]). Long-range dependence seems also an important feature that explains the well-documented evidence of volatility persistence and momentum effects [1,13]. Hereafter we shall only consider the case $H \in (1/2, 1)$, which is most frequently encountered in the real financial data.

The groundwork of modeling the effects of transaction costs was done by Leland [20]. He adopted the hedging strategy of rehedging at every timestep, δt. That is, every δt the portfolio is rebalanced, whether or not this is optimal in any sense. In the following proportional transaction cost option pricing model, we follow the other usual assumptions in the Black–Scholes model but with the following exceptions:

(i) The price S_t of the underlying stock at time t satisfies a fractional Black–Scholes model

$$S_t = S_0 \exp(\mu t + \sigma B_H(t)), \tag{2.1}$$

where $\mu, H \geq \frac{1}{2}, \sigma$ and $S_0 > 0$ are constants.

(ii) The portfolio is revised every δt, where δt is a finite and fixed, small timestep.

(iii) Transaction costs are proportional to the value of the transaction in the underlying. Let k denote the round trip transaction cost per unit dollar of transaction. Suppose υ shares are bought ($\upsilon > 0$) or sold ($\upsilon < 0$) at the price S_t, then the transaction cost is given by $\frac{k}{2}\,|\upsilon|\,S_t$ in either buying or selling, where k is a constant. The value of k will depend on the individual investor. In the fractional Black–Scholes model where transaction costs are incurred at every time the stock or the bond is traded, the no-arbitrage argument used by Black and Scholes no longer applies. The problem is that due to the infinite variation of the geometric fractional Brownian motion, perfect replication incurs an infinite amount of transaction costs.

(iv) The hedged portfolio has an expected return equal to that from an option. This is exactly the same valuation policy as earlier on discrete hedging with no transaction costs.

(v) Traditional economics assumes that traders are rational and maximize their utility. However, if their behavior is assumed to be bounded rational, the traders' decisions can be explained both by their reaction to the past stock price, according to a standard speculative behavior, and by imitation of other traders' past decisions, according to common evidence in social psychology. It is well known that the delta-hedging strategy plays a central role in the theory of option pricing and that it is popularly used on the trading floor. Based on the availability heuristic proposed by Tversky and Kahneman [25], traders are assumed to follow, anchor, and imitate the Black–Scholes delta-hedging strategy to price an option.

A random function $h(x)$ is said to be $O\,(g(x))$, if there exists a constant $M > 0$ such that $\left|\frac{h(x)}{g(x)}\right| \leq M$ a.s. as x is small enough.