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Deconstructing plane anisotropic elasticity Part II: Stroh's formalism sans frills

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Abstract

Eigensolutions for all types of anisotropic elastic materials are obtained in terms of the eigenvalues and the anisotropic elastic stiffness. The generalized eigenvectors and eigensolutions in the degenerate and extra-degenerate cases are obtained by the derivative rule. A complete set of *unnormlized* eigenvectors, now given in terms of the elastic moduli, define the Barnett–Lothe tensors by the same expressions irrespective of material degeneracy. Explicit expressions of the Barnett–Lothe tensors are obtained in various forms depending on the multiplicity of eigenvalues. These expressions complement the alternative expressions of Part I in terms of the elastic compliances. A new family of extra-degenerate materials is found, suggesting the superabundance of such materials. A concise proof of the equivalence of the eigensystems of the compliance-based and elasticity-based formalisms is given. Eigenrelations applicable to all cases of material degeneracy are presented in both three-dimensional and six-dimensional matrix formalisms. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Lekhnitskii (1963) showed that the general solutions of plane anisotropic elasticity may be represented by analytic functions of the complex variables $x + \mu_i y$, where the μ_i's are the roots of a characteristic equation depending on the elastic compliances. The representation allows the differential equations governing the eigensolutions of displacements and stress potentials to be reduced to algebraic equations for the corresponding eigenvectors. An alternative analysis approach, developed by Stroh (1958) and others, yields the eigensolutions in terms of the anisotropic elastic moduli. While the results given in the

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earlier works were restricted to the case when the six roots of the characteristic equation are all distinct, later work by Ting and Hwu (1988) presents the eigensolutions for materials that are degenerate but not extra-degenerate.

In Part I of this paper, a systematic procedure was developed to obtain explicit expressions of the eigensolutions for all types of anisotropic materials, whether nondegenerate, degenerate or extra-degenerate. When the number of independent eigenvectors is smaller than the multiplicity of an eigenvalue, additional eigensolutions must be found in terms of *generalized* eigenvectors and the latter are determined by eigenrelations different from those governing the usual eigenvectors. The generalized eigenvectors and eigensolutions may be obtained by the derivative rule, as shown in Part I for the Lekhnitskii formalism and in the following for the Stroh formalism.

In Part II of this paper, the eigensolutions and the Barnett–Lothe tensors for all types of anisotropic materials are obtained in terms of the elastic moduli, i.e., the elements of $[\beta]^{-1}$. In this approach, the displacement eigenvectors (the **a**-vectors) are to be determined from a 3×3 eigenmatrix $\Gamma(\mu)$ whose elements are quadratic functions of the eigenvalue μ. In contrast, the approach in Part I was based on a 2×2 eigenmatrix $\mathbf{M}(\mu)$ governing the last two components of the **b**-vectors. This asymmetry in the dual formalism contributes to differences in the algebraic analysis and in the formal expressions of the results. However, the two formalisms yield identical eigenvalues and equivalent systems of eigenvectors and generalized eigenvectors. It is shown, as in Part I, that *unnormalized* eigenvectors and generalized eigenvectors satisfy (modified) orthogonality and closure relations, so that the Barnett–Lothe tensors may be defined in the same manner regardless of material degeneracy. Identities involving these tensors, some well known and others new, are shown to be the direct consequences of such definitions, and, therefore, are also valid regardless of material degeneracy.

2. Eigenrelations in the dual formalism

An eigensolution for the displacement, stress potential, strain and stress is given by the following expressions in terms of a complex eigenvalue μ, a pair of eigenvectors $\mathbf{a} = \{a_1, a_2, a_3\}$ and $\mathbf{b} = \{b_1, b_2, b_3\}$, and an arbitrary analytic function f:

$$\mathbf{u} = \mathbf{a}f(x+\mu y), \qquad \mathbf{q} = \boldsymbol{b}f(x+\mu y)$$

$$\{\epsilon\} = \mathbf{E}(\mu)\mathbf{a}f'(x+\mu y), \qquad \{\sigma\} = \sum \mathbf{P}(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} f'(x+\mu_i y), \tag{1}$$

where $\mathbf{u} = \{u, v, w\}^{\mathrm{T}}$, $\{\mathbf{q}\} = \{F_y, -F_x, \Psi\}^{\mathrm{T}}$, $\{\epsilon\} = \{\epsilon_x, \epsilon_y, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^{\mathrm{T}}$, $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^{\mathrm{T}} = \{F_{yy}, F_{xx}, -\Psi_x, \Psi_y, -F_{xy}\}^{\mathrm{T}}$ and

$$\mathbf{E}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 1 \\ \mu & 1 & 0 \end{bmatrix}, \quad \mathbf{P}(\mu) = \begin{bmatrix} -\mu^2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & \mu \\ \mu & 0 \end{bmatrix} \tag{2a,b}$$

The first two components of the **b**-vector are related by $b_1 = -\mu b_2$ because $\tau_{xy} = -\partial_x F_y = -\partial_y F_x$. Substituting the third and fourth expressions of Eq. (1) into the anisotropic stress–strain relation $\{\epsilon\} = [\beta]\{\sigma\}$, one obtains the eigenrelation