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Continuous time Black–Scholes equation with transaction costs in subdiffusive fractional Brownian motion regime

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ABSTRACT

In this paper, we study the problem of continuous time option pricing with transaction costs by using the homogeneous subdiffusive fractional Brownian motion (HFBM) $Z(t) = X(S_\alpha(t))$, $0 < \alpha < 1$, here $dX(\tau) = \mu X(\tau)(d\tau)^{2H} + \sigma X(\tau)dB_H(\tau)$, as a model of asset prices, which captures the subdiffusive characteristic of financial markets. We find the corresponding subdiffusive Black–Scholes equation and the Black–Scholes formula for the fair prices of European option, the turnover and transaction costs of replicating strategies. We also give the total transaction costs.

1. Introduction

The classical and still most popular model of the market is the Black–Scholes model based on the diffusion process call geometric Brownian motion (GBM) [1,2]

$$dX(\tau) = \mu X(\tau)d\tau + \sigma X(\tau)dB(\tau), \quad X(0) = X_0 \tag{1}$$

where μ, σ are constants, and $B(\tau)$ is the Brownian motion. In the presence of transaction costs (TC), Leland [3] first examined option replication in a discrete time setting, and pose a modified replicating strategy, which depends upon the level of transaction costs and upon the revision interval, as well as upon the option to be replicated and the environment. Since then, a lot of authors study this problem, but all in a discrete time setting [4–11].

The option pricing theory as developed by Black–Scholes [1,2] rests on an arbitrage argument: by continuously adjusting a portfolio consisting of a stock and a risk-free bond, an investor can exactly replicate the returns to any option on the stock. It leads us naturally to pose the following question.

In the presence of transaction costs, is there an alternative replicating strategy depending upon the level of transaction costs and a technique leading to the Black–Scholes equation in a continuous time setting? Does the perfect replication incur an infinite amount of transaction costs?

The Black–Scholes (BS) model is based on the diffusion process called geometric Brownian motion (GBM). However, the empirical studies show that many characteristic properties of markets cannot be captured by the BS model, such as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, etc. Therefore, in recent years one observes many generalizations of the BS model based on the ideas and methods known from statistical and quantum physics [12].

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In this paper, we deal with the asset price exhibiting subdiffusive dynamics, $Z(t) = X(S_\alpha(t)), 0 < \alpha < 1$, in which the price of an asset $X(\tau)$ follows a stochastic differential equation

$$dX(\tau) = \mu X(\tau)(d\tau)^{2H} + \sigma X(\tau)dB_H(\tau), \tag{2}$$

where $B_H(\tau)$ is the fractional Brownian motion (FBM) with Hurst exponent $H \in (0, 1)$, and $S_\alpha(t)$ is the inverse α-stable subordinator defined as below

$$S_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}, \tag{3}$$

$U_\alpha(\tau)$ is a strictly increasing α-stable *Lévy* process [13,14] with Laplace transform given by $E(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$. When $\alpha \uparrow 1$, $S_\alpha(t)$ reduces to the "objective time" t. Here, we apply the subdiffusive mechanism of trapping events in order to describe financial data exhibiting periods of constant values as in Ref. [15].

From the Appendix, we can express the model (2) into the following form

$$\frac{dX(\tau)}{d\tau} = \int_0^\tau \mu X(s)2H(2H-1)(\tau-s)^{2H-2}ds + \sigma X(\tau)\xi(\tau) \tag{4}$$

where $\xi(\tau)$ is the fractional Gaussian noise, heuristically $\xi(\tau) = dB_H(\tau)/d\tau$. The FBM has two unique properties: self-similarity and stationary increments [16]. The autocorrelation function of fractional Gaussian noise is the memory kernel $K(\tau), (\tau > 0)$

$$K(\tau) = 2\mathrm{Cov}(\xi(0), \qquad \xi(\tau)) = 2H(2H-1)\tau^{2H-2}, \tag{5}$$

which is rightly the fractional operator in (4). This model was first proposed by Kou to simulate the fluctuation of the distance between a fluorescein–tyrosine pair within a single protein complex [17]. Eq. (4) can be converted to an equation for the time correlation function $C_x(\tau) = E(X(0)X(\tau))$ of $X(\tau)$ by multiplying $X(0)$ and taking expectation, yields

$$\frac{\partial C_x(\tau)}{\partial \tau} = \int_0^\tau \mu K(\tau-s)C_x(s)ds + \sigma E[\xi(\tau)X(\tau)X(0)]. \tag{6}$$

The last term $E[\xi(\tau)X(\tau)X(0)] = 0$ for $\xi(\tau)$ is orthogonal to $X(\tau)$ in the phase space [18]. The Laplace transform of Eq. (6) gives

$$\hat{C}_x(k) = C_x(0)\frac{k^{2H-1}}{k^{2H}-a}, \quad a = \Gamma(2H+1)\mu.$$

Thus, inverting the Laplace transform, we have

$$C_x(\tau) = C_x(0)E_{2H,1}(a\tau^{2H}), \tag{7}$$

where $E_{\alpha,\beta}(t)$ is the Mittag-Leffler function [19]. So, by computing the covariance of the real data, then using $C_x(\tau)$ to approximate it, we can get the value of H. From the correlation function (7), we can get the model is long-dependent for $1/2 < H < 1$. Noting that, the self-similarity, stationary increments and long dependence are all important properties in financial market.

This paper is organized as follows. In Section 2, by using a delta hedging strategy initiated by Leland [3], we deduce the Black–Scholes equation with transaction costs in continuous time setting for the asset price $Z(t) = X(S_\alpha(t)), 0 < \alpha < 1$, where $X(\tau)$ follows (2) and $S_\alpha(t)$ is defined in (3). In the Section 3, we obtain the corresponding Black–Scholes formula. In Section 4, we estimate turnover and transaction costs of replicating strategies. In Section 5, we give the total transaction costs.

2. Continuous time Black–Scholes equation with TC

Theorem 1 (*Continuous Time BS Equation with Transaction Costs in Subdiffusive Regime*)**.** *Let $Z(t) = X(S_\alpha(t))$ be the subdiffusive HFBM, here the parent process $X(\tau)$ is defined in (2), and $S_\alpha(t), 0 < \alpha < 1$, is the inverse α-stable subordinator defined in (3). Assume that the process $S_\alpha(t)$ and $B_H(\tau)$ are independent. Then, the price $V(t, z)$ of a derivative on the subdiffusive FBM stock price $Z(t)$ with transaction cost parameter k is given by the Black–Scholes equation*

$$\frac{\partial V(t,z)}{\partial t} + rz\frac{\partial V(t,z)}{\partial z} + \frac{\tilde{\sigma}^2_{H,\alpha}(t)}{2}z^2\frac{\partial^2 V(t,z)}{\partial z^2} - rV(t,z) = 0, \tag{8}$$

where the modified volatility $\tilde{\sigma}_{H,\alpha}(t)$ is defined, respectively, by

$$\tilde{\sigma}^2_{H,\alpha}(t) = 2Ht^{2H-1}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{2H}\left(1 + \frac{k}{2}\left|\frac{2\mu}{\sigma^2} - 1\right|\mathrm{sign}(\Gamma)\right)\sigma^2 \tag{9}$$

and $\Gamma = \frac{\partial^2 V(t,z)}{\partial z^2}$.