Physica A 390 (2011) 1623–1634

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Scaling and long-range dependence in option pricing V: Multiscaling hedging and implied volatility smiles under the fractional Black–Scholes model with transaction costs ☆

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ARTICLE INFO

Article history:
Received 9 December 2009
Received in revised form 10 April 2010
Available online 24 December 2010

Keywords:
Anchoring adjustment
Delta hedging
Scaling
Implied volatility smiles
Transaction costs

ABSTRACT

This paper deals with the problem of discrete time option pricing using the fractional Black–Scholes model with transaction costs. Through the 'anchoring and adjustment' argument in a discrete time setting, a European call option pricing formula is obtained. The minimal price of an option under transaction costs is obtained. In addition, the relation between scaling and implied volatility smiles is discussed.

1. Introduction

Over the last few years, the financial markets have been regarded as complex and nonlinear dynamic systems. A series of studies has found that many financial market time series display scaling laws and long-range dependence. Therefore, it has been proposed that one should replace the Brownian motion in the classical Black–Scholes model [1] by a process with long-range dependence. A simple modification is to introduce fractional Brownian motion (fBm) as the source of randomness. Thus one adds one parameter, H, to model the dependence structure in the stock prices (for references to these studies see [2–8]). The fractional Black–Scholes model is a generalization of the Black–Scholes model, which is based on replacing the standard Brownian motion by a fractional Brownian motion in the Black–Scholes model. Since fractional Brownian motion is not a semimartingale [9], it has been shown that the fractional Black–Scholes model admits arbitrage in a complete and frictionless market [4–7,9]. However, Guasoni [8] has proved that proportional transaction costs of any positive size eliminate arbitrage opportunities from the fractional Black–Scholes model, but he did not give any corresponding option pricing formulas. Therefore, in a more realistic situation of transaction costs, the magnitude of arbitrage returns associated with those trading strategies in [4–7,10] may create an illusion of profit opportunity when, in fact, none exists.

In this paper, on the basis of the points of view of behavioral finance [11,12] and econophysics [13] and empirical findings of the long-range dependence in stock returns in [14–27], we will study the option pricing problem under transaction costs while the dynamics of stock price S_t satisfies

$$S_t = S_0 \exp\left(\mu t + \int_0^t \sigma_\tau \mathrm{d}B_H(\tau)\right),$$

where $\mu, H \in \left(\frac{1}{2}, 1\right)$, and $S_0 > 0$ are constants; $\sigma_t > 0$ and σ_t is a deterministic and continuous function of time t; and $\int_0^t \sigma_\tau \mathrm{d}B_H(\tau)$ is a pathwise integral with respect to the fBm $B_H(\tau)$.

☆ This work is supported by the National Natural Science Foundation of China (11071082, 10771075).
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0378-4371/$ – see front matter
doi:10.1016/j.physa.2010.12.021

Leland [28] was the first who examined option replication in the presence of transaction costs in a discrete time setting. From the point of view of Leland [28], in a model where transaction costs are incurred every time the stock or the bond is traded, the arbitrage-free argument used by Black and Scholes [1] no longer applies. The problem is that, due to the infinite variation of the geometric Brownian motion, perfect replication incurs an infinite amount of transaction costs. Hence, he suggested a delta-hedging strategy incorporating transaction costs based on revision at a discrete number of times. Transaction costs lead to the failure of the no-arbitrage principle and the continuous-time trade in general: instead of no arbitrage, the principle of hedge pricing – according to which the price of an option is defined as the minimum level of initial wealth needed to hedge the option – comes to the fore.

Mandelbrot (for more details, see the discussions in [2,29]) proposed the trading time concept and considered the problem of choosing the appropriate time scaling to use for analyzing financial market data and pricing options. In Section 2, by using a delta-hedging strategy, initiated by Leland [28], we will show that the price of European options with transaction costs under the fractional Black–Scholes model are determined by the trading time intervals which vary with respect to time t. In Section 3, from the point of view of behavioral finance, we give an explanation for the implied volatility smile in option pricing. In Section 4, a conclusion is given.

2. A European option pricing model for a fractional economy under transaction costs

Let $\left(\Omega, F, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$ be a complete probability space carrying a fractional Brownian motion $(B_H(t))_{t\in R}$ with Hurst exponent $H \in (0, 1)$, i.e., a continuous, centered Gaussian process with covariance function [2]

$$\text{cov}\,(B_H(t)B_H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad s, t \in R;$$

and

$$B_H(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left[\int_{-\infty}^{0}\left[|t-\tau|^{H-\frac{1}{2}} - |\tau|^{H-\frac{1}{2}}\right]\mathrm{d}B(\tau) + \int_0^t |t-\tau|^{H-\frac{1}{2}}\,\mathrm{d}B(\tau)\right], \quad t \in R,$$

where $B(t)$ is a standard Brownian motion process, and, for $t < 0$, the notation $\int_0^t$ should be interpreted an $-\int_t^0$. Here, we assume that $\mathcal{F} = \mathcal{F}_T$ for some $T \in (0, +\infty)$, the sub-σ-algebras $\mathcal{F}_t = \mathcal{B}\left(B_H(\tau), \tau \in [0, t]\right)$ satisfy the usual assumptions of right continuity and saturatedness, $\mathcal{F}_0$ is trivial, and P is the real-world probability measure.

It can be easily seen that $E\,(B_H(t) - B_H(s))^2 = |t-s|^{2H}$. Furthermore, $B_H(t)$ has stationary increments and is H-self-similar; that is, for all $a > 0$, $B_H(at)_{t\in R}$ has the same distribution as $\left(a^H B_H(t)\right)_{t\in R}$; and, for all $\delta t > 0$, $(B_H(t+\delta t) - B_H(t))_{t\in R}$ has the same distribution as $\left[a^{-H}\,(B_H(t+a\delta t) - B_H(t))\right]_{t\in R}$.

If $H = \frac{1}{2}$, then the corresponding fractional Brownian motion is the usual standard Brownian motion.

If $H > \frac{1}{2}$, the process $(B_H(t),\ t \geq 0)$ exhibits a long-range dependence; that is, if $r(n) = E\left[B_H(1)\,(B_H(n+1) - B_H(n))\right]$, then $\sum_{n=1}^{\infty} r(n) = \infty$. As mentioned in [27], long-range dependence is widespread in economics and finance and has remained a topic of active research (e.g., see [2] for details). Long-range dependence seems also an important feature that explains the well-documented evidence of volatility persistence and momentum effects. If $H < \frac{1}{2}$, the fractional Brownian motion is said to be anti-persistent: disjoint increments are negatively correlated.

It has been shown that a Hurst exponent $H \neq \frac{1}{2}$ does not necessarily imply long-time correlations like those found in the fractional Brownian motion. However, according to Pan's arguments [30], price momentum is analogical to positive autocorrelation in stock returns, which could arise because of an investor's underreaction or continuing overreaction to news. Therefore, the fractional Black–Scholes model appears to be consistent with the behavioral models which suggest positive autocorrelation in stock returns in the short run.

Hereafter we shall only consider the case $H \in \left(\frac{1}{2}, 1\right)$, which is most frequently encountered in real financial data (e.g., see [14–26], where empirical evidence is given of the Hurst exponent with values in $\left(\frac{1}{2}, 1\right)$ for financial data and the applicability of the fractional Brownian motion model to particular financial markets is also obtained).

Since the fractional Brownian motion is not a semimartingale for $H \neq \frac{1}{2}$, we cannot use the martingale theory to define stochastic integrals with respect to it. Many authors [4–7,31–37] have proposed basically two types of integral: pathwise and Skorohod (Wick–Itô) integrals. In particular, under the Skorohod (Wick–Itô) integral frame, Norwegian mathematicians have obtained many interesting original results on the fractional Black–Scholes model and given a proof to the existence of Itô-like formulas (e.g., see [31–37] for details). However, there is a lot of controversy about its economic interpretation even if the introduction of Wick–Itô calculus in the fractional Black–Scholes model has inspired some promising results [38]. In the following, we give a simple introduction to the pathwise integral with respect to $B_H(t)$ with $H > \frac{1}{2}$ (e.g., see [4–7]).

Assume that $\phi(t) = \phi\,(t, \omega)$ is left-continuous with right limits. Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[0, T]$. Put $\delta_n = \max_{0\leq i\leq n-1}\{t_{i+1} - t_i\}$ and define $\int_0^T \phi(t)\mathrm{d}B_H(t) = \lim_{\delta_n\to 0}\sum_{i=0}^{n-1}\phi\,(t_i)\cdot(B_H\,(t_{i+1}) - B_H\,(t_i))$, if the limit exists in probability.

In the continuous-time fractional Black–Scholes model, if the stock price process S_t is given by

$$S_t = S_0\exp(\mu t + \sigma B_H(t)), \tag{2.1}$$