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Dynamic advertising and pricing with constant demand elasticities

Kurt L. Helmes *, Rainer Schlosser

Humboldt-Universität zu Berlin, Berlin, Germany

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ABSTRACT

In this paper we analyze a stochastic dynamic advertising and pricing model with isoelastic demand. The state space is discrete, time is continuous and the planing horizon is allowed to be finite or infinite. A dynamic version of the Dorfman–Steiner identity will be derived. Explicit expressions of the optimal advertising and pricing policies, of the value function and of the optimal advertising expenditures will be given. The general results will be used to analyze the case of impatient customers. Furthermore, particular time inhomogeneous models and homogeneous ones with and without discounting will be examined. We will study the social efficiency of a monopolist's optimal policies and the consequences of specific subsidies. From a buyer's perspective, our analysis reveals that waiting – when looking at (immediate) expected prices – is never profitable should two or more units be available. But we will also prove that the sequence of average sales prices is monotone decreasing. Moreover, the techniques applied to solve the discrete stochastic advertising and pricing problem will be used to solve a related deterministic control problem with continuous state space.

1. Introduction

The theory of dynamic pricing of stochastic models in the general context of revenue management is well established, see Talluri and van Ryzin (2004) and the review papers Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003) and Shen and Su (2007); for additional references see Bitran and Mondschein (1993), McAfee and te Velde (2006), and the literature cited therein. The theory of how to set optimal advertising rates together with optimal prices when selling perishable products in a stochastic environment is yet still in its infancy, see MacDonald and Rasmussen (2009) for an excellent literature revue.

However, there is an extensive literature on deterministic advertising models and combined advertising and pricing models, for instance, Nerlove and Arrow (1962), Bass (1969), and many others; see the surveys Mahajan et al. (1990), Feichtinger et al. (1994), and Bagwell (2007) for details and annotated bibliographies. These models typically consider the sale of durable goods, the introduction of new goods and the controlled evolution of, for example, the market share of a product with the objective to maximize overall profits. The Sethi (1983) model, is a well researched example of a deterministic model and a stochastic model of such kind. The state variable of the Sethi model represents the market share of a product which is driven by Brownian motion and controlled by the rate of advertising effort. The variation of the

* Corresponding author. Tel.: +49 30 20935744; fax: +49 30 20935745.
E-mail addresses: helmes@wiwi.hu-berlin.de (K.L. Helmes), schlosser.rainer@freenet.de (R. Schlosser).

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original model considered in Sethi et al. (2008) is an example of a deterministic model where the state is controlled by the marketing instruments advertising and price.

At first glance the traditional revenue management problems, e.g. selling airplane tickets of a particular flight, and classical advertising models, e.g. increasing the market share of a newly introduced product, seem to deal with rather different kind of situations. It will be pointed out in Section 7 that several aspects of the well known models of both camps are but two sides of the same coin. Differences exist in the way state spaces are described and interpreted, the horizon of the control problems, whether or not unit costs are taken into account and, very important, the modeling of the stochastic environment. In the context of classical advertising models stochastic driving forces are often modeled by Brownian motion or more general stochastic diffusion processes. In revenue management the Poisson process and more general jump Markov processes play a prominent role.

For perishable products, MacDonald and Rasmussen (2009) is a recent paper where simultaneous advertising and pricing decisions are analyzed in a stochastic environment. The stochastic environment is due to the random arrival of customers and their willingness to pay. MacDonald and Rasmussen consider the case when N units of a perishable product are to be sold over a finite horizon T. Their model assumes the initial stock of the product to decrease according to a Poisson type process with time homogeneous intensity λ. For each price value $p \geq 0$ and advertising rate $w \geq 0$ the intensity has the form $\lambda(p,w) = a \cdot w^{\delta} \cdot e^{-mp}$, $0 \leq \delta < 1$, where a and m are positive constants. The factor aw^{δ} reflects the (average) number of shoppers which are attracted by advertising to take a look at a product. In the pure pricing version of the model the average number of shoppers is assumed to be equal to the fixed number a. The second factor reflects the force (depending on price) which turn shoppers into buyers. Their intensity function λ is a combination of the classical willingness to pay function considered in Gallego and van Ryzin (1994) and the increasing concave function of the advertising rate with constant elasticity δ. For their model, which is an extension of the pure pricing model studied by Gallego and van Ryzin, MacDonald and Rasmussen derive closed form expressions of the optimal advertising and pricing policies as well as the value function. The value function is defined as the maximum of expected revenue minus expected advertising cost of all policies under consideration. The authors also evaluate a fixed-price, fixed-advertising heuristic which can be easily implemented and can also be used when the demand is stochastic. Their numerical analyses show the performance of this advertising and pricing heuristic for the model considered in MacDonald and Rasmussen (2009).

Gallego and van Ryzin (1994) analyze dynamic pricing problems with general reservation price distributions for deterministic models as well as for stochastic ones. In particular, see Theorem 4.1 of Gallego and van Ryzin (1994), they prove the value function of a deterministic pricing problem to dominate the value function of the corresponding stochastic problem. In the special case of exponential demand (i.e. $\lambda(p) = ae^{-mp}$), they prove that optimal prices of a stochastic model (given in feedback form) are larger than the optimal prices of the corresponding deterministic problem, cf. Proposition 3 of Gallego and van Ryzin (1994). This property also holds in the case with advertising if demand is isoelastic, see below. For isoelastic models we derive analytical expressions of the value function of a deterministic problem and its stochastic counterpart. These analytical formulas make it possible to precisely evaluate the difference between deterministic and expected profits.

In this paper we shall look at the situation when the time dependent jump intensity λ of the process of unsold units – a pure "death process" – is of the form $\lambda(t,p,w) = a(t) \cdot w^{\delta} \cdot p^{-\varepsilon}$ where $a(t) > 0$ is a given deterministic function, $\varepsilon > 1$ and $0 \leq \delta < 1$. This intensity has two important characteristics. Since p is allowed to be any positive number, even arbitrarily small, the factor $p^{-\varepsilon}$ of λ implies that all items will be sold over any finite time interval $[t_0, T]$, $0 \leq t_0 < T$, if the objective is to maximize the expected revenue and no unit cost term is taken into account. Moreover, not only is the advertising elasticity of λ constant, but the price elasticity is constant as well.

The assumption of constant elasticities has far reaching implications. In the case of a stochastic (pure) pricing problem it has been pointed out by McAfee and te Velde (2008) that constant price elasticity of demand and zero unit costs imply a monopolist to set socially efficient prices. In the deterministic case, Stiglitz (1976) has actually shown that under the same assumptions the optimal pricing strategy coincides with the (perfectly) competitive price. The combination of isoelastic demand and zero (unit) cost implies that optimal sales intensities may be unbounded, and in the deterministic case as well as the stochastic case any initial stock will be always cleared over any given period. If advertising is endogenized, i.e. $\delta > 0$, we will prove that this property still holds true but that a monopolist will not choose socially efficient prices. However, in Section 6, we will describe mechanisms which lead to efficient policies by a monopolist. In particular, an adjustment which is a combination of a revenue tax and an advertising subsidy will be proposed which guarantees efficient prices and which is (even) self-financing.

To analyze the problem we extend and generalize results obtained in McAfee and te Velde (2008) for the pure pricing model to the case with advertising. The dynamic (pure) pricing model has many applications, see Talluri and van Ryzin (2004) for a variety of examples. Applications of models which combine advertising and pricing include, for instance, one-time events which are individually advertised. Examples include ticket sales of singular sporting events (Olympic Games, Champions League soccer games, etc.), special concerts, shows and performances. Fish and fruit markets are other well known examples where combined advertising and pricing activities can be observed almost every day. Not only do prices typically drop close to the end of the sales period but the volume and the frequency of the "shouting" of sellers increases as well. From a general perspective, these examples are special cases of the situation when a decision maker is selling a given number of identical (or very similar) assets over a fixed period of time, and advertising as well as discounting is common. Specific industries include the business of new car dealers at the end of a (car)model year, or sectors of the real estate