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On the exploitation of chaos to build reduced-order models

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Abstract

The present study focuses on the model reduction of non-linear systems. The proper orthogonal decomposition is exploited to compute eigenmodes from time series of displacement. These eigenmodes, called the proper orthogonal modes, are optimal with respect to energy content and are used to build a low-dimensional model of the non-linear system. For this purpose, the proper orthogonal modes obtained from a chaotic orbit are considered. Indeed, such an orbit is assumed to cover a portion of the phase space of higher dimension, and hence of greater measure. This higher dimensional data is further assumed to contain more information about the system dynamics than data of a lower-dimensional periodic orbit. In an example, it is shown that the modes for this particular behaviour are more representative of the system dynamics than any other set of modes extracted from a non-chaotic response. This is applied to a buckled beam with two permanent magnets and the reduced-order model is validated using both qualitative and quantitative comparisons.

Keywords: Model reduction; Non-linear dynamics; Proper orthogonal decomposition; Chaos

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1. Introduction

In many domains of applied sciences and in structural dynamics particularly, dealing with large-scale dynamical structures is a central issue. In the presence of non-linearities, seeking for the solution by use of mathematical modelling and simulation (e.g., finite element method) may be computationally intensive. Accordingly, due to the complexity of such a numerical approach, it is worth reducing the dimensionality of the system while retaining its intrinsic properties.

The general philosophy of model reduction is to find a co-ordinate transformation in order to sort the components in terms of their influence on the system behaviour. Then, the components of the transformed system with relatively small influence may be truncated without substantially degrading the predictive

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capability of the model. The proper orthogonal decomposition (POD), also known as Karhunen–Loève transform or principal component analysis (PCA), enables such a co-ordinate transformation. It is a statistical pattern analysis technique for finding the dominant structures in an ensemble of spatially distributed data. These structures, called the proper orthogonal modes (POMs), may be exploited as an orthogonal basis for efficient representation of the ensemble. A key advantage of the decomposition is that each POM is associated with a proper orthogonal value (POV) which provides the relative energy captured by the corresponding mode. Thus, it serves as a well-defined measure of a mode influence on the system behaviour.

The present study is motivated by the fact that, in the field of non-linear systems, new features must be defined because mode shapes are no longer effective to represent the system dynamics. While some similarities between the POMs and the modes shapes have been noticed [1,2], the POMs are much more useful for capturing the dynamics of a non-linear system. In Ref. [3], lower-dimensional models of non-linear vibrating systems are created using the POMs in order to prove their efficiency and their superiority over the mode shapes.

In this work, the POMs obtained from a chaotic orbit are considered. Indeed, such an orbit is assumed to cover a portion of the phase space of higher dimension, and hence of greater measure. This higher dimensional data is further assumed to contain more information about the system dynamics than data of a lower-dimensional periodic orbit. Additionally, if the dimension of the data is d, then the number of states needed to describe the data is bounded by $2d + 1$, and also leads to a higher number of identifiable modes. The correlation between the dimension of the active phase space and the number of significant proper orthogonal modes has been observed (e.g., in [4]). In the example studied here, it is shown that the modes for this particular behaviour are more representative of the system dynamics than any other set of modes extracted from a non-chaotic response. This is applied to a buckled beam with two permanent magnets and the reduced-order model is validated using both qualitative and quantitative comparisons.

2. Proper orthogonal decomposition

The POD was proposed independently by several scientists including Karhunen [5], Kosambi [6], Loève [7], Obukhov [8] and Pougachev [9] (see Ref. [10] for a recent survey) and was originally conceived in the framework of continuous second-order processes. When restricted to a finite dimensional case and truncated after a few terms, the POD is equivalent to PCA. This latter methodology originated with the work of Pearson [11] as a means of fitting planes by orthogonal least squares and was also proposed by Hotelling [12].

The first applications of the POD in the field of structural dynamics date back to the early 1990s. Cusumano et al. [4] exploited the technique to estimate the intrinsic dimensionality of the dynamics of an impacting beam. Kreuzer and Kust [13] used it to control self-excited vibrations of long torsional strings. Kappagantu and Feeny [14] worked on the modal reduction of a frictionally excited system. Other studies include the works of Azeez and Vakakis [15], Benguedouar [16], Fitzsimons and Rui [17], Georgiou and Schwartz [18] and Lenaerts et al. [19].

The central idea of the POD is to reduce a large number of interdependent variables to a much smaller number of uncorrelated variables while retaining as much as possible of the variation in the original variables. An orthogonal transformation to the basis of the eigenvectors of the sample covariance matrix is performed and the data is projected onto the subspace spanned by the eigenvectors corresponding to the largest eigenvalues. This transformation decorrelates the signal components and maximises variance. The most striking property of the POD is its optimality in the sense that it minimises the average squared distance between the original signal and its reduced representation.

For the sake of brevity, the complete mathematical description of the POD is not recalled here. The reader is referred to Ref. [20] for a detailed description. For practical applications, the data is discretised in