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Mean-square Riemann–Stieltjes integrals of fuzzy stochastic processes and their applications

Yuhu Feng *

Department of Basic Sciences, China Textile University, Shanghai, 200051, China

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Abstract

The mean-square Riemann–Stieltjes integrals of two types associated with a class of fuzzy stochastic processes are defined. The integrability conditions, some of integral formulas and properties are established. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction and notation

The study of fuzzy stochastic theory has been developed in recent years (e.g. [1–4] and their references). From the standpoint of applications, the mean-square theory of fuzzy stochastic calculus and fuzzy stochastic differential equations is attractive because of its simplicity and because of its applications to practical problems follow, in broad outline, the corresponding ordinary deterministic procedures. In [2] we have developed the concepts of mean-square Riemann integral and differential associated with a class of fuzzy stochastic processes and studied their integrability and differentiability properties. Riemann–Stieltjes integrals in the mean square can be developed parallel to the development of mean-square Riemann integrals as in the case of real valued stochastic processes. Our objective in this paper is to consider mean-square Riemann–Stieltjes integrals of the types

$$\mathrm{I} = \int_a^b X(t)\,\mathrm{d}g(t) \quad \text{and} \quad \mathrm{II} = \int_a^b g(t)\,\mathrm{d}X(t),$$

where $X(t)$ is a second-order fuzzy stochastic processes (see Definition 2.3 of [2]) and $g(t)$ is a real function defined on $[a,b]$. As expected, the mean-square integrals I and II exist for a class of real functions and second-order fuzzy stochastic processes and they also possess many formal properties of ordinary Riemann–Stieltjes integrals. In the non-random case the integrals I and II degenerate into the Riemann–Stieltjes integrals of a

* Tel.: +86 21 6237 3331; fax: +86 21 6219 4722.
E-mail address: basdep@ctu.edu.cn (Y. Feng)

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fuzzy valued function we have never seen which were discussed before. Furthermore, for the important case of Gaussian fuzzy stochastic process, the Gaussian property is preserved under the mean-square Riemann–Stieltjes integrals. In practice when many problems are only concerned in the mean and the second-order moment of some fuzzy stochastic process, it is of benefit in solving problems to regard the process as a Gaussian fuzzy stochastic process. One of the main difficulties in works of this paper, especially in discussion on the integral II, is the fact that the fuzzy number space (E^n, D) is not a linear space. It would be possible to overcome this by using the concept of the H-difference of fuzzy numbers.

We first recall for the convenience of the reader some notations on the fuzzy number space (E^n, D) and the second-order fuzzy random variable space (L^2, ρ).

Let $E^n = \{u : R^n \to [0,1] \mid u$ satisfies (i)–(iv) below$\}$, where (i) u is normal, i.e. there exists an x_0 such that $u(x_0) = 1$; (ii) u is fuzzy convex, i.e. $u(rx + (1-r)y) \geqslant \min(u(x), u(y))$, $x, y \in R^n$, $r \in [0, 1]$; (iii) u is upper semicontinuous; (iv) $[u]^0 = \overline{\{x \in R^n \mid u(x) > 0\}}$ is compact. If $A \in \mathscr{P}_c(R^n)$, the family of all nonempty compact convex subsets of R^n, I_A is its characteristic function, then $I_A \in E^n$. A linear structure in E^n is defined as usual. Let $u, v \in E^n$, and set

$$D(u,v) = \sup_{0 \leqslant r \leqslant 1} d([u]^r, [v]^r),$$

where $[u]^r = \{x \in R^n \mid u(x) \geqslant r\}$, $0 < r \leqslant 1$, is the r-level set of u, d is the Hausdorff metric defined in $\mathscr{P}_c(R^n)$, i.e.

$$d(A,B) = \max\left(\sup_{a \in A} \inf_{b \in B} |a-b|, \sup_{b \in B} \inf_{a \in A} |a-b|\right),$$

for all $A, B \in \mathscr{P}_c(R^n)$, where $|\cdot|$ denotes the usual Euclidean norm in R^n. The norm $\|u\|$ of a fuzzy number $u \in E^n$ is defined by

$$\|u\| = D(u, \hat{0}) = \|[u]^0\| = \sup_{a \in [u]^0} |a|,$$

where $\hat{0}$ is the fuzzy number in E^n which membership function equals 1 at 0 and zero elsewhere.

(E^n, D) is a complete metric space [4].

Let $(\Omega, \mathscr{A}, P)$ be a complete probability space. A fuzzy random variable (f.r.v. for short) is a Borel measurable function $X : (\Omega, \mathscr{A}) \to (E^n, D)$. If $E\|X\| < \infty$, then the expected value EX exists.

Let $L^2 = \{X \mid X$ is an f.r.v. with $E\|X\|^2 < \infty\}$. Two f.r.v.'s X and Y are called equivalent if $P(X \neq Y) = 0$. Then all equivalent elements in L^2 are identified. Define

$$\rho(X,Y) = \left(ED^2(X,Y)\right)^{1/2}, \quad X, Y \in L^2.$$

The norm $\|X\|_2$ of an element $X \in L^2$ is defined by

$$\|X\|_2 = \rho(X, \hat{0}) = (E\|X\|^2)^{1/2}.$$

(L^2, ρ) is a complete metric space [2, Corollary 2.2] and ρ satisfies that

$$\rho(X+Z, Y+Z) = \rho(X,Y), \ \rho(\lambda X, \lambda Y) = |\lambda| \rho(X,Y), \tag{1.1}$$

$$\rho(\lambda X, kX) \leqslant |\lambda - k| \, \|X\|_2, \tag{1.2}$$

for any $X, Y, Z \in L^2$ and $\lambda, k \in R$.

Let $u, v \in E^n$. If there exists a $w \in E^n$ such that $u = v + w$ then we call w the H-difference of u and v, denoted by $u \ominus v$. Because $u \ominus v : E^n \times E^n \to E^n$ is continuous, if X and Y are f.r.v.'s and the H-difference of X and Y exists a.s., i.e. $P(X \ominus Y \text{ exists}) = 1$, then $X \ominus Y$ is an f.r.v. and $X \ominus Y \in L^2$ provided $X, Y \in L^2$.