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A Nonlinear Dynamics Perspective of Moment Closure for Stochastic Processes

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1 Introduction

Many partial differential equations (PDE's) in applied mathematics can be reduced to an infinite set of ordinary differential equations (ODE's) using the Fourier transform or similar. Each such ODE describes the evolution of a single Fourier coefficient, or from an alternative point of view the evolution in one co-ordinate direction in an infinite dimensional function space. This approach can often simplify the analysis and numerical solution of such equations. Examples include paradigm systems such as the Complex Ginzburg Landau Equation or the Navier Stokes Equation with periodic boundary conditions, *eg* [1,2]. Of course, no computer can represent the full infinite set of ODE's, and hence when we apply this idea to the numerical solution of PDE's, we necessarily have to truncate the set to a finite number of equations. Classically this was done by ignoring all higher order Fourier coefficients beyond a certain point, giving a so called *Galerkin approximation*.

A very similar approach has evolved in the theory of some stochastic processes. Many such processes can be described by the time evolution of a *generating function* $Q(t,z)$. A variety of such generating functions can be defined, depending on whether their coefficients (Fourier or Taylor) yield probabilities, moments, cumulants *etc*, see for instance [3]. Again, one can replace the PDE for Q by an infinite set of ODE's for the moments or cumulants, often facilitating its solution. Although for some simple stochastic processes it may be possible to solve the complete set explicitly, for more complex systems it is again necessary to ignore higher order coefficients, typically by setting all those above some given order to 0, *eg* [3]. This is thus entirely analogous to the Galerkin method.

Within the last decade an important class of generalizations of the classical Galerkin method has appeared, motivated by the theory of so called *inertial manifolds, eg* [1,2]. In the current context an inertial manifold M yields a function Ψ that gives each high order Fourier coefficient as a function of a finite set of low order coefficients (Figure 1). All trajectories asymptotically approach M, so that after transients have died out the dynamics of the system is given exactly by the finite dimensional system lying on M. The classical Galerkin method corresponds to $\Psi = 0$, whilst so called *nonlinear Galerkin methods*, [2,4,5] correspond to using an exact or approximate Ψ (see below).

At the same time, it has been noticed [6-10] that for some stochastic processes good approximations can be obtained by assuming that the probability distribution is restricted to some given class of distributions, *eg* normal, negative binomial, *etc.* In the case of the normal distribution, such approximations date back to [11]. Although in some cases a theoretical justification can be given (*eg* see references in [7]), in general it remains somewhat of a mystery why this procedure can be so successful. Observe, however, that assuming a particular distribution is often equivalent to assuming a functional relation amongst the moments, which in turn can be expressed as a function giving higher order moments in terms of the first few. Hence this procedure is essentially equivalent to a nonlinear Galerkin method

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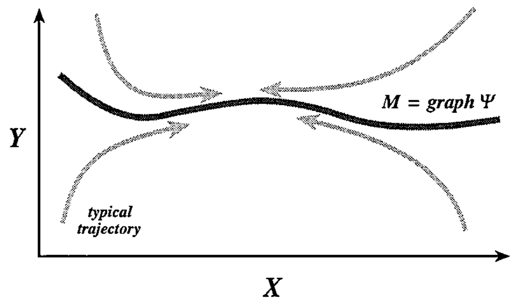

Figure 1 The inertial manifold M is the graph of the function $\Psi: X \to Y$, where X is the finite dimensional subspace spanned by the first few Fourier coefficients and Y is the infinite dimensional space given by the remaining coefficients. M itself is invariant under the dynamics.

and its success suggests that the original system may possess an inertial manifold. The aim of this paper is to investigate this possibility, and more generally to explore the relevance of inertial manifold ideas to stochastic processes. As a particular example we shall consider the stochastic model of a host-parasite interaction introduced by Isham [8]. This has the advantage that it is explicitly solvable and hence allows us to easily compute the various approximations that we present and compare them to the exact solution of the system.

2. Generating Functions

The Isham model [8] describes the evolution of the probability distribution of the number of parasites in a single host, or more precisely the conditional probability given that the host is still alive. In abstract terms it therefore specifies a time dependent probability measure on the non-negative integers. If we denote the probability of $\{j\}$ at time t by $p_j(t)$, we can define, *eg* [3], the *probability generating function* $Q(t,z)$ by

$$Q(t,z) = \sum_{j=0}^{\infty} p_j(t) z^j \tag{1}$$

Since the p_j are probabilities, their sum must be exactly 1, and hence $Q(t,1) = 1$. Thus the radius of convergence of Q is at least 1, *ie* (1) converges for at least $|z| \leq 1$. The p_j can be recovered from Q by differentiating j times at $z = 0$. The evolution of Q can often be described conveniently by a differential equation

$$\frac{\partial Q}{\partial t} = F(Q) \tag{2}$$

on an appropriate function space. The moments m_k are defined, *eg* [3], by

$$m_k(t) = \sum_{j=0}^{\infty} p_j(t) j^k$$

assuming that this converges. They can be derived from the probability generating function using the *moment generating function* $\tilde{Q}(t,\theta) = Q(t,e^{\theta})$. Assuming that Q is sufficiently regular (in particular that the radius of convergence of Q is strictly greater than 1), it is easy to verify that