ELSEVIER

Physica A 300 (2001) 300–309

PHYSICA A

www.elsevier.com/locate/physa

# Time evolution of stochastic processes with correlations in the variance: stability in power-law tails of distributions

Boris Podobnik[a,b,*], Kaushik Matia[b], Alessandro Chessa[b,c], Plamen Ch. Ivanov[b], Youngki Lee[b,d], H. Eugene Stanley[b]

[a] *Department of Physics, Faculty of Sciences, University of Zagreb, Zagreb, Croatia*

[b] *Center for Polymer Studies and Department of Physics, Boston University, Boston, MA 02215, USA*

[c] *Dipartimento di Fisica and Unitá INFM, Universitá di Cagliari, 09124 Cagliari, Italy*

[d] *Yanbian University of Science & Technology, Beishan St., Yanji, Jilin 133000, China*

Received 1 August 2001

**Abstract**

We model the time series of the S&P500 index by a combined process, the AR+GARCH process, where AR denotes the autoregressive process which we use to account for the short-range correlations in the index changes and GARCH denotes the generalized autoregressive conditional heteroskedastic process which takes into account the long-range correlations in the variance. We study the AR+GARCH process with an initial distribution of truncated Lévy form. We find that this process generates a new probability distribution with a crossover from a Lévy stable power law to a power law with an exponent outside the Lévy range, beyond the truncation cutoff. We analyze the sum of $n$ variables of the AR+GARCH process, and find that due to the correlations the AR+GARCH process generates a probability distribution which exhibits stable behavior in the tails for a broad range of values $n$—a feature which is observed in the probability distribution of the S&P500 index. We find that this power-law stability depends on the characteristic scale in the correlations. We also find that inclusion of short-range correlations through the AR process is needed to obtain convergence to a limiting Gaussian distribution for large $n$ as observed in the data. © 2001 Elsevier Science B.V. All rights reserved.

*Keywords:* Random walks; Stochastic processes; Fluctuation phenomena; Central limit theory

## 1. Introduction

Application of statistical physics methods to analyze probability distribution functions (PDFs) of financial data has attracted recent interest. Much work [1–21] has been

* Corresponding author.


PII: S0378-4371(01)00390-9

devoted to determine precisely the functional form of these PDFs. For the S&P500 stock index, it has been shown [7,21] that the central profile of the PDF is well described by the Lévy distribution [22]. Recent analyses [21] of high frequency data have shown that the PDF is described by a crossover to a power law, with exponent $1+\alpha$ well beyond the Lévy range $(0<\alpha<2)$. (The data cover the period 1 January 1985 to 31 December 1995, and the time interval between successive records of the index is 1 min.) The tails of the PDF appear to exhibit stability for long but finite time scales.

In addition to the form of PDFs, other important, but complementary related quantities, are the absolute value and variance of price changes, which are commonly used as measures of the risk [3]. For the S&P500 index, in contrast to time series of price changes that show only short-range correlations [9,21,23], the time series of absolute values of price changes exhibit long-range correlations [23–27]. It is natural to ask how slow-decaying (long-range) correlations in the variance may be related to the scaling behavior observed in PDFs.

To describe the S&P500 index, we analyze the generalized autoregressive conditional heteroskedastic (GARCH) process [28] to take account of correlations in the variance of price changes. (A process is autoregressive if variable $x_t$ depends on its own lagged values. Heteroskedasticity is related with non-constant variances.) In addition to the GARCH process, we employ the autoregressive (AR) process [29] to account for the effect of the short-range correlations in price changes and develop a combined process, the AR+GARCH process. We show that due to the GARCH process, the AR+GARCH process generates the power-law tails in the PDF with an exponent outside the Lévy range. (Power-law tails in distributions can be obtained in multiplicative processes introduced in Refs. [30] and [31].) The GARCH process itself is constructed out of independent and identically distributed (i.i.d.) stochastic variables specified by arbitrary PDF. With the choice of truncated Lévy PDF [32], we model the crossover behavior in the PDF of the AR+GARCH process as observed in the data [21]. We study a process that is the sum of $n$ AR+GARCH variables to probe for large $n$ the stability of the PDF. For this new process, we find long-range correlations in the variance arising from the GARCH process. We also identify the long-range correlations as the source of the empirically observed stability for a range of time scales in the power-law tails of the PDF.

## 2. The GARCH process for a truncated Lévy distribution

### 2.1. The GARCH process

Let us define an independent and identically distributed (i.i.d.) stochastic variable $v_t$ with zero mean ($\langle v_t \rangle = 0$) and unit variance ($\langle v_t^2 \rangle = 1$). The generalized autoregressive conditional heteroskedastic (GARCH) process $x_t$ [28] is a discrete time stochastic