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Research note

Exact closed-form solutions for "optimal inventory model for items with imperfect quality and shortage backordering"

Hung-Chi Chang [a], Chia-Huei Ho [b,*]

[a] Department and Graduate School of Logistics, National Taichung Institute of Technology, Taichung 404, Taiwan
[b] Graduate School of Management, Ming Chuan University, No. 250 Zhong Shan North Road, Section 5, Taipei 111, Taiwan

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ABSTRACT

Wee et al. [Optimal inventory model for items with imperfect quality and shortage backordering. Omega 2007;35(1):7–11] recently contributed an optimal inventory model for items with imperfect quality and shortage backordering. This article revisits their study and applies the well-known renewal-reward theorem to obtain a new expected net profit per unit time. We derive the exact closed-form solutions to determine the optimal lot size, backordering quantity and maximum expected net profit per unit time, specifically without differential calculus. We also solve the same model algebraically from another direction, which has been mentioned, but the process has not been finished yet. The problem parameter effects upon the optimal solutions are examined analytically and numerically.

1. Introduction

The classical economic order/production quantity (EOQ/EPQ) models, although well known and useful, implicitly assume that all products have perfect quality, which may not conform to real situations since defective items often exist in order/production batches. Consequently, many researchers developed inventory/production models to include defective items and/or imperfect production process, to study the effects of imperfect quality on the optimal inventory/production policies. Porteus [1] and Rosenblatt and Lee [2] were the first to address the imperfect production process problems. Their works have encouraged many researchers to explore quality related issues, see e.g. [3–13]. Specifically, Wee et al. [9] extended Salameh and Jaber's [3] EOQ model for items with imperfect quality to include shortages. Both of them [3,9] assumed that each lot received or produced contains a random fraction of imperfect quality items. The defective items are picked up as a single batch within the replenishment period, where a 100% lot screening process is conducted at a fixed rate. Items of poor quality are sorted, kept in stock and sold at a salvage value prior to receiving the next shipment. In a recent article, Maddah and Jaber [11] enhanced [3] by applying renewal theory (see, e.g. Ross [14]) to obtain the expected profit per unit time. They indicated that since the process generating the profit is a renewal process, hence it is more appropriate to measure the expected profit function using renewal-reward theory.

The classical optimization technique based on differential calculus undoubtedly is a powerful and useful method to solve inventory decision models. The above mentioned studies used it to find the optimal solutions or derive the conditions for optimality, but it is replaceable. Grubbström [15] first derived the classical EOQ formula algebraically. Grubbström and Erdem [16] extended this approach to solve EOQ model with backlogging. These two studies received considerable attention and invoked many researchers to propose various algebraic methods to solve inventory related models with or without shortages (see, e.g., [17–22]). A comprehensive review of the literatures up to 2006 can be found in Cárdenas-Barrón [23], who further classified the algebraic procedure according to its difficulty level as simple, medium and high. Recent studies proposed other distinct approaches. For example, Teng [24] suggested the arithmetic geometric mean inequality (AM-GM) theorem, Hsieh et al. [25] provided the Cauchy–Schwarz inequality and Wee et al. [26] modified the cost-difference comparison method. Besides, Chang et al. [27] pointed out a direction for further research to algebraically solve an EPQ model with shortages. To the best of our knowledge, this open problem has not yet been solved. Moreover, we note that in the most recent studies Pentico et al. [28] and Zhang [29] also proposed the new and/or alternative approaches to solve the EPQ model with partial backordering.

In this article, we shall revisit Wee et al.'s [9] model (an optimal inventory model for items with imperfect quality and shortage backordering). As in Maddah and Jaber [11], we first apply the renewal-reward theorem to derive new expected net profit per unit time. Without using differential calculus, we derive the exact closed-form solutions for optimal lot size and maximum shortage level, as well as the expected profit per unit time. We also propose an approach to finish the task left in Chang et al. [27].

* Corresponding author. Tel.: +886 2 28824564x2131.
E-mail addresses: hungchi@ntit.edu.tw (H.-C. Chang), chho@mail.mcu.edu.tw (C.-H. Ho).

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2. A brief review of Wee et al.

For the sake of clarity and to make the analysis tractable, this section briefly reviews the works done by Wee et al. [9]. The notations used in [9] and this paper are listed in Appendix. As mentioned earlier, [9] extended Salameh and Jaber's [3] model to allow shortages, where the environments are as described in the previous section. Fig. 1 shows the behavior of this inventory model. Since the batch of products contains imperfect quality, it is implicitly assumed that even though the 100% screening process has not been conducted while receiving the products, the backordered products will be delivered without any defects. This assumption is missing in [9], but is necessary for Wee et al.'s [9] model and this study.

Based on the assumptions and notations in [9], the total revenue (total sales volume of good quality and the imperfect quality items) per cycle is expressed as

$$TR(p,y)=(1-p)ys+pyv. \tag{1}$$

The total cost (ordering, purchasing, screening, holding, and backordering) per cycle is expressed as

$$TC(B,y)=K+cy+dy+h\left[\frac{(y-py-B)^2}{2D}+\frac{py^2}{x}\right]+\frac{bB^2}{2D}. \tag{2}$$

The net profit per unit time is then measured using $TPU(B,y)=[TR(p,y)-TC(B,y)]/T$, where $T=(1-p)y/D$. The objective function, the expected net profit per unit time, denoted by $ETPU(B,y)$, is derived by taking the expected value of $TPU(B,y)$, i.e., $ETPU(B,y)=E[TPU(B,y)]$.

For the case that the defective percentage p is distributed uniformly, using the classical optimization technique based on differential calculus, Wee et al. [9] derived the optimal conditions to solve (B^*,y^*) (the optimal value of (B,y)) and proved $ETPU(B,y)$ is concave. They then indicated that using software such as Maple can solve (B^*,y^*) simultaneously.

3. Recast Wee et al.'s model

This section recasts Wee et al.'s [9] model using the renewal-reward theorem. The new expected net profit per unit time (denoted by $NEPTU(B,y)$) is derived as follows:

$$\begin{aligned}NEPTU(B,y)&=\frac{E[TR(p,y)]-E[TC(B,y)]}{E(T)}=Ds+Dv\frac{E(p)}{E(1-p)}-\frac{D(c+d)}{E(1-p)}\\&\quad-\left\{\frac{DK}{y}\frac{1}{E(1-p)}+\frac{bB^2}{2y}\frac{1}{E(1-p)}\right.\\&\quad\left.+\frac{h}{2}\frac{E[(1-p)^2]}{E(1-p)}y-hB+\frac{hB^2}{2y}\frac{1}{E(1-p)}+\frac{hyD}{x}\frac{E(p)}{E(1-p)}\right\}.\end{aligned} \tag{3}$$

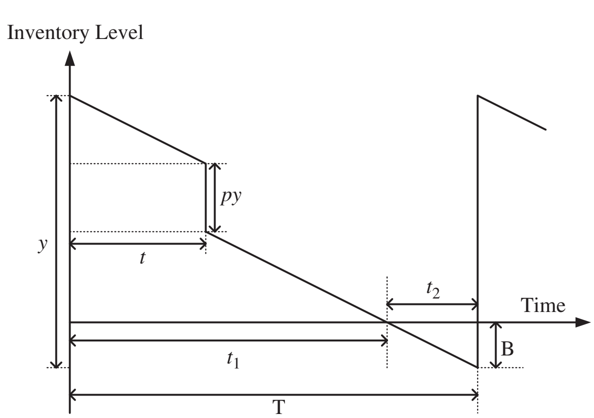

Fig. 1. Inventory system with complete backordering [9].

Since the first three terms in Eq. (3) are independent of decision variables (B,y), they can be dropped in determining (B^*,y^*). Consequently, maximizing $NEPTU(B,y)$ is equivalent to minimizing the cost terms in brace (we denote it using $f(B,y)$). For notational convenience, let

$$P_1=\frac{1}{E(1-p)},P_2=\frac{E[(1-p)^2]}{E(1-p)},P_3=\frac{E(p)}{E(1-p)}, \tag{4}$$

then $f(B,y)$ is written as

$$f(B,y)=\frac{DK}{y}P_1+\frac{bB^2}{2y}P_1+\frac{h}{2}P_2y-hB+\frac{hB^2}{2y}P_1+\frac{hyD}{x}P_3. \tag{5}$$

Instead of using the classical optimization technique, this study provides two possible directions to derive the exact closed-form solutions for optimal (B^*,y^*) and minimum cost $f(B^*,y^*)$ without derivatives.

Approach 1

Similar to Chang et al. [27], Eq. (5) can be rewritten as

$$\begin{aligned}f(B,y)&=\frac{(h+b)P_1}{2y}\left[B-\frac{hy}{(h+b)P_1}\right]^2\\&\quad+\frac{h}{P_1}\left[P_1P_2+\frac{2DP_1P_3}{x}-\frac{h}{(h+b)}\right]\frac{y}{2}+\frac{DKP_1}{y}.\end{aligned} \tag{6}$$

Thus, for given y, setting the square term in Eq. (6) to zero results in

$$B=\frac{hy}{(h+b)P_1}, \tag{7}$$

and Eq. (6) reduces to

$$f(y)=\frac{h}{P_1}\left[P_1P_2+\frac{2DP_1P_3}{x}-\frac{h}{(h+b)}\right]\frac{y}{2}+\frac{DKP_1}{y}. \tag{8}$$

Next, from Teng [24], using the arithmetic geometric mean inequality (AM-GM) theorem,

$$f(y)\geq\sqrt{2hDK\left\{P_1P_2+\frac{2DP_1P_3}{x}-\frac{h}{(h+b)}\right\}} \tag{9}$$

When the two terms related to y in Eq. (8) are equal, implying

$$y^*=\sqrt{\frac{2DK}{h\left\{\frac{P_2}{P_1}+\frac{2DP_3}{xP_1}-\frac{h}{(h+b)P_1^2}\right\}}}, \tag{10}$$

and Eq. (9) reduces to equality, that is, the minimum cost is

$$f^*=\sqrt{2hDK\left\{P_1P_2+\frac{2DP_1P_3}{x}-\frac{h}{(h+b)}\right\}}. \tag{11}$$

Once y^* is obtained, B^* follows from Eq. (7).

To get more insight of solutions, we further express $P_i, i=1,2,3$ with exact forms. Without specifying the probability density function of p, we consider that the mean μ_p and variance σ_p^2 of p are known. For any particular distributions, the mean and variance can be measured easily or found from Statistics textbooks. By Eq. (4), $P_1=1(1-\mu_p)$, $P_2=[(1-\mu_p)^2+\sigma_p^2]/(1-\mu_p)$, $P_3=\mu_p/(1-\mu_p)$, then Eqs. (10), (11), (7) and (3) reduce to

$$y^*=\sqrt{\frac{2DK}{h\left[\frac{b(1-\mu_p)^2}{(h+b)}+\frac{2D\mu_p}{x}+\sigma_p^2\right]}}, \tag{12}$$