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Near-optimal (r,Q) policies for a two-stage serial inventory system with Poisson demand ☆

Lei Yang [a,*], Jian Yang [b], Gang Yu [c], Hanqin Zhang [d,e]

[a] School of Economics and Commerce, South China University of Technology, Guangzhou, Guangdong 510006, PR China
[b] Department of Mechanical and Industrial Engineering, New Jersey Institute of Technology, Newark, NJ 07102, USA
[c] The Store Corporation, Shanghai 201203, PR China
[d] Academy of Mathematics and Systems Science, The Chinese Academy of Sciences, Beijing 100080, PR China
[e] NUS Business School, Singapore

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ABSTRACT

We consider a two-stage serial inventory system whose cost structure exhibits economies of scale in both stages. In the system, stage 1 faces Poisson demand and replenishes its inventory from stage 2, and the latter stage in turn orders from an outside supplier with unlimited stock. Each shipment, either to stage 2 or to stage 1, incurs a fixed setup cost. We derive important properties for a given echelon-stock (r, Q) policy for an approximation of the problem where all states are continuous. Based on these properties, we design a simple heuristic algorithm that can be used to find a near-optimal (r, Q) policy for the original problem. Numerical examples are given to demonstrate the effectiveness of the algorithm.

1. Introduction

We are concerned with the management of a two-echelon stochastic inventory system involving setup costs. Research on multi-echelon inventory systems started about four decades ago, when Clark and Scarf (1960) published their seminal paper that characterized the optimal control policy when only the leading stage dealing with a limitless supplier is charged with setup costs. When economies of scale appear in more than one stage, however, the characterization of optimal policies meets much tougher resistances. Subsequent research on multi-echelon stochastic inventory systems with economies of scale focused more on heuristic policies. For more information about these heuristics, the reader is referred to Axaster (1993a,b), Chen and Zheng (1994a,b), Federgruen and Zipkin (1984), Shang (2008), Chao and Zhou (2009), Olsson (2009), Hariga (2010), Shang and Zhou (2010), as well as the comprehensive survey papers by Chen (1999b) and Axsater (2003).

We study the following two-stage serial inventory system. Stage 1 directly faces Poisson demand and replenishes its inventory by ordering from stage 2, which in turn orders from an outside supplier with unlimited stock. Economies of scale are reflected in the ordering costs at both stages. That is, each shipment, either to stage 2 or to stage 1, incurs a fixed setup cost. We limit our search for the optimal policy to the (r, Q) type. Formally, stage 2 orders a fixed quantity Q_2 from its immediate upper stream whenever its echelon inventory position (outstanding orders plus on-hand inventory at stage 2 and the entire inventory at or in-transit to all down-stream stages minus backorders at stage 1) drops to a re-order point r_2. Stage 1 places an order with quantity Q_1 whenever its inventory position (inventory in-transit plus on-hand inventory minus backorder) drops to r_1. Policies of this type are widely used in practice because the more restricted order sizes accommodate easy packaging, transportation, and coordination.

We make the simplifying assumption that the leadtime from the outside supplier to stage 2 is zero. This is reasonable when stage 2 has a strong bargaining power over the outside supplier and the latter is physically close by. In deterministic inventory systems, this assumption enabled the development of 98%-effective policies (see, e.g., Roundy, 1985, 1986). In stochastic inventory systems, this assumption has been adopted by numerous researchers, including Naddor (1956, 1963), Pressman (1977), Nahmias and Demmy (1981), Azoury and Brill (1986), and Hill (1999). Also under this assumption, Chen (1999a) established a lower bound for the long-run average cost of any feasible policy, and then provided a 94%-effective heuristic policy. The zero-supplier-leadtime assumption is known for two important consequences:

(1) The optimal policy is nested, in the sense that stage 2 must send a shipment to stage 1 whenever it receives an order.

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* Corresponding author. Fax: +86 20 39382288.
E-mail address: yang@scut.edu.cn (L. Yang).

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(2) The optimal policy possesses the zero-inventory-ordering property; that is, stage 2 must have zero on-hand inventory when it orders.

We first demonstrate important properties of any given (r,Q) policy when applied to the continuous-state approximation of the original problem, and then achieve the optimization of policy parameters by reducing this approximated problem to two single-variable quasi-convex optimization problems. Next, based on the above properties, we design a simple heuristic algorithm that can be used to determine near-optimal policy parameters for the original problem. Our numerical experiments attest to the effectiveness of this algorithm. For most problems tested, the algorithm reaches 99%-effectiveness, and shows great improvements over Chen's method.

The rest of the paper is organized as follows. We present the two-stage model as well as some known results in Section 2. In Section 3, we present properties revolving around the echelon-stock (r,Q) policy in the continuous-state approximation of the original model. In Section 4, we provide a heuristic algorithm that can be used to compute near-optimal policy parameters for the original problem. We present some numerical examples in Section 5, and conclude the paper in Section 6. All mathematical proofs have been relegated to Appendix.

2. Problem formulation and existing results

Here we give a more detailed description of the two-stage inventory system. Demand arrives at stage 1 according to a Poisson process. Stage 1 is replenished by stage 2, which in turn orders from an outside supplier with unlimited stock. Each shipment, either to stage 2 or to stage 1, incurs a fixed setup cost. The leadtime from the outside supplier to stage 2 is zero and the leadtime from stage 2 to stage 1 is nonnegative. Unsatisfied demand at stage 1 is fully backlogged. The planning horizon is infinite, and the objective is to minimize the long-run average system-wide cost. Let

λ be the Poisson demand arrival rate;
K_i be the fixed setup cost at stage i, $i=1,2$;
D be the total demand during the leadtime from stage 2 to stage 1;
h_i be the echelon holding cost rate at stage i, $i=1,2$;
p be the backorder cost rate at stage 1.

Since the leadtime at stage 2 is zero, we can save the holding cost at stage 2 by delaying ordering at this stage until just before the next shipment to stage 1. Therefore, an optimal policy must be nested, so that stage 2 will send a shipment to stage 1 immediately following the former's receipt of a shipment from the outsider supplier. Under a nested policy, suppose stage 2 places an order at time 0. Then, the order will be received immediately and a shipment will be sent to stage 1 at time 0. We call this the first shipment from stage 2 to stage 1. Let

n be the total number of shipments from stage 2 to stage 1 between two consecutive orders at stage 2;
r_1^{i-1} be stage 1's inventory position just before the ith shipment from stage 2 to stage 1, $i=2,\ldots,n$;
r_1^n be stage 1's inventory position right before stage 2 places an order;
r_2 be stage 2's echelon inventory position right before it places an order;
Q_2 be the total demand between two consecutive orders at stage 2;
Q_1^i be the total demand between the ith and $(i+1)$th shipment from stage 2 to stage 1, $i=1,\ldots,n-1$;
Q_1^n be the total demand between the time when stage 2 sends the last shipment to stage 1 and the time when stage 2 places the next order.

If we use R_1^i to denote the size of the ith shipment from stage 2 to stage 1, $i=1,\ldots,n$, then we have

$$R_1^i = \begin{cases} r_1^1+Q_1^1-r_1^n, & i=1, \\ r_1^i+Q_1^i-r_1^{i-1}, & 1<i\le n. \end{cases} \quad (1)$$

Thus,

$$Q_2 = \sum_{i=1}^{n} R_1^i = (r_1^1+Q_1^1-r_1^n)+\sum_{i=2}^{n}(r_1^i+Q_1^i-r_1^{i-1}) = \sum_{i=1}^{n} Q_1^i.$$

Furthermore, because the installation inventory level at stage 2 is always nonnegative, we have $r_2 \ge r_1^n$.

Following Chen (1999a), we can express the long-run average system-wide cost as

$$C(r_2,Q_2,\mathbf{r}_1,\mathbf{Q}_1) = \frac{\lambda K_2 + \sum_{y=r_2+1}^{r_2+Q_2} h_2 y + \sum_{i=1}^{n}(\lambda K_1 + \sum_{y=r_1^i+1}^{r_1^i+Q_1^i} G_1(y))}{Q_2},$$

where $\mathbf{r}_1=(r_1^1,\ldots,r_1^n)$, $\mathbf{Q}_1=(Q_1^1,\ldots,Q_1^n)$, and

$$G_1(y) = \mathrm{E}[h_1(y-D)^+ + (p+h_2)(y-D)^-]. \quad (2)$$

We omit detailed derivations here.

For tractability concerns, we take the continuous-state approximation; see, e.g., Zheng (1992) and Chen (1999a). Once all decision variables are supposed to be continuous, we get the following continuous-state approximation of our original problem:

$$\text{Problem } \mathbf{P}: \begin{cases} \min & [\lambda K_2 + \int_{r_2}^{r_2+Q_2} h_2 y\,\mathrm{d}y + \sum_{i=1}^{n}(\lambda K_1 \\ & + \int_{r_1^i}^{r_1^i+Q_1^i} G_1(y)\,\mathrm{d}y)]/Q_2 \\ \text{s.t.} & \sum_{i=1}^{n} Q_1^i = Q_2, \\ & r_1^n < r_1^1+Q_1^1, \quad \text{and } r_1^i < r_1^{i+1}+Q_1^{i+1} \text{ for } i=1,\ldots,n-1, \\ & r_1^n = r_2, \\ & Q_2>0, \quad \text{and } Q_1^i>0 \\ & \text{for } i=1,\ldots,n, \\ & n\ge 1, \text{ integer.} \end{cases}$$

In problem $\mathbf{P}$, the first constraint says that the total demand between two consecutive orders at stage 2 is the sum of the demands in the n disjoint intervals between consecutive shipments from stage 2 to stage 1. The second constraint is present because the size of each shipment must be strictly positive; see (1). We have made the third constraint an equality instead of the more anticipated inequality $r_1^n \le r_2$. Suppose r_1^n were strictly below r_2. Then, since $\int_{r_2}^{r_2+Q_2} h_2 y\,\mathrm{d}y = h_2(2r_2Q_2+Q_2^2)/2$ is increasing in r_2, we are able to reduce the objective value by replacing r_2 by the strictly smaller r_1^n while keeping all other variables intact. This shows that, at optimality, stage 2 should carry no inventory $(r_2-r_1^n=0)$ upon ordering. The last two constraints merely stem from the definition of the decision variables.

For any $Q>0$ and r, define

$$C_1(r,Q) = \frac{\lambda K_1 + \int_r^{r+Q} G_1(y)\,\mathrm{d}y}{Q},$$

where $K_1>0$ is the fixed setup cost at stage 1 and $G_1(y)$ is given by (2). We may interpret $C_1(r,Q)$ as the expected long-run average cost of the single-stage model with setup cost K_1 and the holding-backorder cost rate function $G_1(y)$ under the given (r, Q) policy; see