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# Some comments on deteriorating inventory theory with partial backlogging

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ABSTRACT

We provide an easier way to prove the main theorems in Dye (2007) and Dye et al. (2007) using a result from nonlinear programming. We also extend the validation of the conclusion in Dye (2007) and Dye et al. (2007) to more cases of the demand function and correct some mistakes on this topic appeared in the literature.



In papers published on Omega (Dye, 2007) and European Journal of Operational Research (Dye et al., 2007), the authors investigate a joint pricing and ordering problem for deteriorating products with time-dependent backlogging rate. The former paper employs the hyperbolic backlogging rate, while the latter uses the exponential backlogging rate. Both papers first prove that given a selling price, there exists a unique optimal replenishment schedule (Theorem 1 in Dye, 2007; Dye et al., 2007). They then show given that the marginal revenue with respect to price is decreasing, the total profit is a concave function of price when the replenishment schedule is given. However, in the numerical example provided in both papers, demand rate $d(p)=16\times 10^7 p^{-3.21}$ does not satisfy the condition that the marginal revenue with respect to price is decreasing since $pd(p)$ is convex.

We first take a look at the proof of Theorem 1 in Dye (2007). We refer readers to Dye (2007) for the definition of the notation used below. The total profit per unit time is given as follows:

$$\begin{aligned}\Pi(p,t_1,t_2)=\frac{1}{t_1+t_2}\{&(p-c)d(p)(t_1+t_2)-K-cd(p)\\&\times\int_0^{t_1}[e^{g(t)}-1]\,dt-hd(p)\int_0^{t_1}e^{-g(t)}\int_t^{t_1}e^{g(u)}\,du\,dt\\&-\frac{[s+\delta(p-c+\pi)]d(p)}{\delta^2}[\delta t_2-\ln(1+\delta t_2)]\},\end{aligned}$$

where $g(z)=\int_0^z\theta(u)\,du$ and $\theta(t)$ is the instantaneous decay rate.

In general, given a twice differentiable function $f(t_1,t_2)$, if $(t_1^*,t_2^*)$ is a stationary point, $\partial^2 f(t_1^*,t_2^*)/\partial t_1\partial t_2=0$ does not hold. For example, $f(t_1,t_2)=t_1^2+2t_2-t_1t_2$. Therefore, the claim

$$\left.\frac{\partial^2\Pi(t_1,t_2|p)}{\partial t_1\partial t_2}\right|_{(t_1^*,t_2^*)}=0,$$

in the proof of part (b) of Theorem 1 is not straightforward. Define $z(p,t_1,t_2)=(t_1+t_2)\Pi(p,t_1,t_2)$. For any given $p$, the first-order conditions of the original problem are

$$\frac{\partial\Pi(t_1,t_2|p)}{\partial t_1}=-\frac{z(t_1,t_2|p)}{(t_1+t_2)^2}+\frac{1}{t_1+t_2}\frac{\partial z(t_1,t_2|p)}{\partial t_1}=0$$

$$\frac{\partial\Pi(t_1,t_2|p)}{\partial t_2}=-\frac{z(t_1,t_2|p)}{(t_1+t_2)^2}+\frac{1}{t_1+t_2}\frac{\partial z(t_1,t_2|p)}{\partial t_2}=0$$

It is easy to see that $\partial^2 z(t_1,t_2|p)/\partial t_1\partial t_2=0$ in this case. Thus,

$$\begin{aligned}\frac{\partial^2\Pi(t_1,t_2|p)}{\partial t_1\partial t_2}=&\frac{2z(t_1,t_2|p)}{(t_1+t_2)^3}-\frac{1}{(t_1+t_2)^2}\frac{\partial z(t_1,t_2|p)}{\partial t_2}\\&-\frac{1}{(t_1+t_2)^2}\frac{\partial z(t_1,t_2|p)}{\partial t_1}\end{aligned}$$

Substituting the first-order conditions into the equation above, we have $\partial^2\Pi(t_1,t_2|p)/\partial t_1\partial t_2|_{(t_1^*,t_2^*)}=0$.

The proof of Theorem 1 in Dye (2007) is very complicated. Here we show that there exists a finite global maximizer for $\Pi(t_1,t_2|p)$ using a different approach.

**Lemma 1.** *For any given p, $z(t_1,t_2|p)$ is a concave function of $(t_1,t_2)$.*

**Proof.**

$$\begin{aligned}\frac{\partial^2 z(t_1,t_2|p)}{\partial t_1^2}=&-cd(p)\theta(t_1)e^{g(t_1)}\\&-hd(p)\theta(t_1)e^{g(t_1)}\int_0^{t_1}e^{-g(t)}\,dt-hd(p)<0\end{aligned}$$

$$\frac{\partial^2 z(t_1,t_2|p)}{\partial t_2^2}=-\frac{[s+\delta(p-c+\pi)]d(p)}{(t_1+t_2)^2}<0$$

Since $\partial^2 z(t_1,t_2|p)/\partial t_1\partial t_2=0$, the Hessian of $z(t_1,t_2|p)$ is negative definite, which completes the proof. □

**Lemma 2.** *Let $\phi$ and $\psi$ be functions defined on a set $\Gamma\subset R^n$ and $\psi\neq 0$ on $\Gamma$. Let $\Gamma$ be open, let $\bar{x}\in\Gamma$, and let $\phi$ and $\psi$ be differentiable at $\bar{x}$, then $\phi/\psi$ is pseudoconcave at $\bar{x}$ if $\phi$ is concave at $\bar{x}$, $\psi>0$, and $\psi$ is linear on $R^n$* (Mangasarian, 1994, p. 148).

**Corollary 1.** *For any given p, $\Pi(t_1,t_2|p)$ is a pseudoconcave function of $(t_1,t_2)$ on $(0,+\infty)\times(0,+\infty)$.*

**Theorem 1.** *If the instantaneous decay rate $\theta(t)$ is an increasing function of time t, then there exists a finite global maximizer for $\Pi(t_1,t_2|p)$.*

**Proof.** Due to the positive backorder and lost sale costs, the optimal $t_2$ cannot be $+\infty$. If $\theta(t)$ increases with $t$, the instantaneous decay rate goes to infinity as $t_1$ goes to infinity, which

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means that the optimal $t_1$ cannot be $+\infty$. However, if the backorder and lost sale costs are large enough, it is not optimal to have backlogged and lost demand, i.e., $t_2^* = 0$. Based on Corollary 1, a finite global maximizer is guaranteed and it is easy to find this global maximizer numerically. □

Dye et al. (2007) investigate the same problem under the assumption that the backlogging rate is exponential and prove that Theorem 1 above holds as well. Similarly, their proof is also very complicated. The authors argue that the optimal length of the shortage period is infinity under certain condition. Based on common sense, if backlog and lost sale costs are positive, it will never be optimal to stay on shortage forever. Abad (2008) tries to generalize Theorem 1 above for any decreasing backlogging rate $B(t)$, where $t$ is the amount of time that the customer waits before receiving the product. Unfortunately, the main result is wrong. In the proof of Proposition 2 in page 184, the author claims that if $p \geq v + c_1 - c_3 > v + c_1 - c_3 - c_2\psi$ then $p - v - c_1 + c_3 - c_2\psi \geq 0$ by mistake.

For general decreasing backlogging rate $B(t)$, the inventory level can be written as

$$I(t) = \begin{cases} d(p)e^{-g(t)} \int_t^{t_1} e^{g(u)}\,du & \text{if } 0 \leq t < t_1, \\ -d(p) \int_{t_1+t_2-t}^{t_2} B(x)\,dx & \text{if } t_1 \leq t < t_1 + t_2. \end{cases}$$

The number of units sold in one cycle is $d(p)t_1 + d(p)\int_0^{t_2} B(x)\,dx$. The order quantity is $d(p)\int_0^{t_1} e^{g(u)}\,du + d(p)\int_0^{t_2} B(x)\,dx$. The inventory holding cost is as same as the holding cost in Dye (2007). Backlog and lost sale costs in one cycle are $sd(p)\int_0^{t_2} xB(x)\,dx$ and $\pi[d(p)t_2 - d(p)\int_0^{t_2} B(x)\,dx]$, respectively. Given $p$, the total profit in one cycle is

$$\begin{aligned} F(t_1,t_2|p) = {} & p\left[d(p)t_1 + d(p)\int_0^{t_2} B(x)\,dx\right] - K \\ & - c\left[d(p)\int_0^{t_1} e^{g(u)}\,du + d(p)\int_0^{t_2} B(x)\,dx\right] \\ & - hd(p)\int_0^{t_1} e^{-g(t)} \int_t^{t_1} e^{g(u)}\,du\,dt - sd(p)\int_0^{t_2} xB(x)\,dx \\ & - \pi d(p)t_2 + \pi d(p)\int_0^{t_2} B(x)\,dx. \end{aligned} \quad (1)$$

Given $p$, the total profit per unit time is

$$\Gamma(t_1,t_2|p) = \frac{F(t_1,t_2|p)}{t_1+t_2}.$$

**Theorem 2.** *If the instantaneous decay rate $\theta(t)$ is an increasing function of time $t$, then there exists a finite global maximizer for $\Gamma(t_1,t_2|p)$.*

**Proof.** The proof of Theorem 1 is based on Corollary 1, which does not hold for $\Gamma(t_1,t_2|p)$. However, we know that

$$\frac{\partial^2 F}{\partial t_1^2} = -cd(p)e^{g(t_1)}\theta(t_1) - hd(p)e^{g(t_1)}\theta(t_1)\int_0^{t_1} e^{-g(t)}\,dt - hd(p) < 0,$$

$$\frac{\partial F}{\partial t_2} = (p - c + \pi - st_2)B(t_2) - \pi,$$

$$\frac{\partial^2 F}{\partial t_2^2} = (p - c + \pi - st_2)\frac{dB}{dt_2} - sB(t_2),$$

$$\frac{\partial^2 F}{\partial t_1 \partial t_2} = 0.$$

When $t_2 \leq (p-c+\pi)/s$, the Hessian of $F(t_1,t_2|p)$ is negative definite, which means that $\Gamma(t_1,t_2|p)$ is pseudoconcave on $(0,+\infty) \times (0,(p-c+\pi)/s]$.

Now we show that the optimal $t_2$ for $\Gamma(t_1,t_2|p)$ satisfies the condition $t_2 \leq (p-c+\pi)/s$. Since

$$\frac{\partial \Gamma}{\partial t_2} = \frac{[(p-c+\pi-st_2)B(t_2)-\pi](t_1+t_2) - F(t_1,t_2|p)}{(t_1+t_2)^2}.$$

If $t_2 > (p-c+\pi)/s$, $[(p-c+\pi-st_2)B(t_2)-\pi](t_1+t_2) < 0$. Now it is easy to see that for any $t_1$ and $t_2 > (p-c+\pi)/s$, $\Gamma(t_1,t_2|p) \leq \max\{\Gamma(t_1,(p-c+\pi)/s|p),0\}$. □

We now turn our attention to the existence of the unique selling price when the replenishment schedule is given, which is shown by Dye (2007) and Dye et al. (2007) when $pd(p)$ is concave and $d(p)$ is convex. However, the example provided, $d(p) = 16 \times 10^7 p^{-3.21}$, does not satisfy previous condition. Abad (2008) proves pseudoconcavity when $1/d(p)$ is convex. However, the assumption of the convexity of $1/d(p)$ is not weaker than the assumption of the concavity of $pd(p)$. For example, when $d(p) = p^{-0.5}$, $pd(p)$ is concave, but $1/d(p)$ is not convex. We now show the existence of the unique selling price for both types of demand functions when the replenishment schedule is given.

**Lemma 3.** *If $f(x)$ is twice differentiable and if the condition*

$$f'(x) = 0 \Rightarrow f''(x) \leq 0$$

*holds, then $f(x)$ is unimodal* (Boyd and Vandenberghe, 2006, p. 101).

**Theorem 3.** *If $1/d(p)$ is a convex function of $p$, or $pd(p)$ is concave and $d(p)$ is convex, then there exists a unique optimal selling price for $\Gamma(p|t_1,t_2) = F(p|t_1,t_2)/(t_1+t_2)$, where $F(p|t_1,t_2)$ has the same expression as $F(t_1,t_2|p)$ in Eq. (1).*

**Proof.** When the replenishment schedule is given, the total profit per unit time can be written as

$$\Gamma(p|t_1,t_2) = pd(p)U(t_1,t_2) - d(p)V(t_1,t_2) - \frac{K}{t_1+t_2},$$

where

$$U(t_1,t_2) = \frac{t_1 + \int_0^{t_2} B(x)\,dx}{t_1+t_2} > 0,$$

$$\begin{aligned} V(t_1,t_2) = {} & \frac{1}{t_1+t_2}\left\{\int_0^{t_1} e^{g(u)}\,du + \int_0^{t_2} B(x)\,dx\right] \\ & + h\int_0^{t_1} e^{-g(t)}\int_t^{t_1} e^{g(u)}\,du\,dt \\ & + s\int_0^{t_2} xB(x)\,dx + \pi\left[t_2 - \int_0^{t_2} B(x)\,dx\right]\right\} > 0 \end{aligned}$$

If $pd(p)$ is concave and $d(p)$ is convex, then $\Gamma(p|t_1,t_2)$ is obviously concave. If $1/d(p)$ is convex, the first-order condition can be written as

$$pd'(p)U(t_1,t_2) + d(p)U(t_1,t_2) - d'(p)V(t_1,t_2) = 0.$$

If $p$ satisfies the first-order condition, then

$$\begin{aligned} \frac{d^2\Gamma(p|t_1,t_2)}{dp^2} & = 2d'(p)U(t_1,t_2) + pd''(p)U(t_1,t_2) - d''(p)V(t_1,t_2) \\ & = \frac{2[d'(p)]^2 - d''(p)d(p)}{d'(p)}U(t_1,t_2) \end{aligned}$$

Since $1/d(p)$ is convex, we have $2[d'(p)]^2 - d''(p)d(p) \geq 0$. Also, $d'(p) < 0$. Therefore, $\Gamma(p|t_1,t_2)$ is unimodal according to Lemma 3. Since demand will be zero if $p$ is large enough, the optimal price is finite. This completes the proof. □