Int. J. Production Economics ∎ (∎∎∎∎) ∎∎∎–∎∎∎

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Adapting inventory models for handling various payment structures using net present value equivalence analysis

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ARTICLE INFO

Article history:
Received 16 October 2012
Accepted 22 September 2013

Keywords:
Deterministic inventory theory
Net present value
Payment structure
Economic order quantity
Economic production quantity
Philosophy of modelling

ABSTRACT

Classic inventory models use average cost functions. It is generally accepted that these models should account for the time value of money. They do so not by considering the timing of cash-flows, but by including opportunity costs. The Net Present Value (NPV) framework has long been used to compare these models with. We formalise NPV Equivalence Analysis (NPVEA) under various payment structures, and apply it to a few classic inventory models. While taking the linear approximation is typically part of the process to find equivalence, the essence is to disregard the parameters of a classic inventory model but instead start from cash-flow structures between firms. It is demonstrated how this leads to different plausible interpretations of, or variations to, classic inventory models, in particular for payment structures that differ from conventional assumptions. We identify situations with negative holding costs, which indicates that more features from the real world must be added into the decision model. We illustrate that in addition to capital costs, firms can enjoy capital rewards. These rewards may not always affect the firm's inventory decisions, but are in general useful for finding the impact of changes to various parameters on the firm's future profits.

1. Introduction

The foundation that inventory theory is to account for the time value of money goes back to Harris (1913), who was careful to explain that his Economic Order Quantity (EOQ) model is largely about the trade-off with the opportunity cost of capital. His is the archetypal model of classic inventory theory, in which one minimises the average costs per unit of time. The holding cost is commonly found from the integration over a relevant cycle time T:

$$\frac{1}{T}\int_0^T h(t)I(t)\,dt, \tag{1}$$

where $I(t)$ is the inventory level at time t, and $h(t)=h$ the unit holding cost, typically taken to be time-invariant. Costs are not discounted according to their time of occurrence, but the time value of money is *implicitly* modelled by the inclusion into h of the financial opportunity cost from stock investment. Typically, h is taken to be of the form (Silver et al., 1998):

$$h=\alpha v+f, \tag{2}$$

where v is the money invested per unit of product held in stock, f the unit 'out-of-pocket' holding cost, and α the firm's continuous capital rate. A value $\alpha=0.20$ (time measured in years) is often used, and the putative view is that the financial holding cost dominates: $\alpha v>>f$.

The opportunity cost is also the foundation for the Net Present Value (NPV), which quite generally can be viewed to be the Laplace transform of a cash-flow function $a(t)$ in which the Laplace frequency is taken to be α (Grubbström, 1967):

$$\int_0^\infty a(t)e^{-\alpha t}\,dt. \tag{3}$$

As the time value of money is modelled explicitly, it would be incorrect to include into $a(t)$ the financial holding cost as used in classic models. Instead, it is retrieved in the linearised Annuity Stream (AS) function (Grubbström, 1980). The AS is the constant payment stream having the same NPV as a given stream of payments; for an infinite horizon, AS $=\alpha$ NPV.

The comparison with NPV has been used at least as early as Hadley and Whitin (1963) and Hadley (1964), who demonstrate that Harris' model retains the lot-size relevant terms of the linear AS function. Grubbström (1980, 2007) shows how capital costs can be determined for inventories and work-in-process at several stages in more complex systems of production and inventory. See also Gurnani (1983). Porteus (1985) clarifies how the timing of expenditures and revenues relative to the cycle time of a

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0925-5273/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.ijpe.2013.09.013

regenerative process affects the valuation of capital costs, and illustrates using Harris' EOQ model. Teunter et al. (2000), Van der Laan and Teunter (2002), Teunter and van der Laan (2002), and Çorbacioğlu and van der Laan (2007) use the linear AS function for setting the unit holding costs at the different stages in systems of remanufacturing. They demonstrate that the mapping of the classic parameters to the AS function is not necessarily injective. Beullens and Janssens (2011) introduce the Anchor Point (AP) in NPV models, and show that its position in the supply chain can affect the valuation of capital costs at the different stages in the system.

In this paper we use NPV to study the impact of *payment structures* on the unit holding cost and other parameters in classic models. Grubbström (1980) was perhaps first to introduce the term. Most studies that use NPV to retrieve the capital costs, except for Porteus (1985) and Beullens et al. (submitted for publication), adopt 'conventional' assumptions. Loosely speaking, this means that costs or revenues are assumed to occur either the moment that some (physical) process in the system initiates or terminates, such as the transfer of a batch of materials, or continuously at the rate of process transformation, such as a production rate. As the timing of payments is not described in classic inventory models, this first guess often leads to satisfactory results. However, trends in inventory management, including the use of consignment stocks and credit delays, show that a wider variety of payment structures are adopted in practise. The question is then under which variety of payment structures the classic models, and their solutions, can still be used, or how they should be adapted.

The study of *equivalence* is formalised as NPVEA in Section 2, and a basic idea from propositional logic introduced. We belief that this makes it easier to present results, and their logical consequences, succintly. In Section 3 payment structures are defined and common examples presented. In Sections 4 to 7, the approach is applied to a few well-known inventory models. Next to finding out the strengths and weaknesses of these models, it leads to a number of simple variations which do not appear in our literature, but should be of relevance in the context of various practical applications. As a general conclusion, we find that the study of equivalence under various payment structures can be of great help to better understand inventory theory, and increase and extend its applicability.

The paper considers infinite horizon models with constant demand, but NPV can also be used for dynamic lot-sizing, see e.g. Grubbström in press. NPV is not the only possible framework to help making financial decisions about the future, see e.g. Xu et al. (2012) for a comparison with real options.

2. NPV equivalence analysis

A satisfactory understanding of a classic inventory model is not automatically arrived at when adopting the NPV viewpoint, as classic inventory theory is insufficient to describe cash-flow functions. The extra degrees of freedom in NPV imply that there is an, in principle infinite, number of possible interpretations of a classic model. Insight into the meaning of a classic model's (cost) parameters is best achieved by not using them in the NPV reference model. Application of NPVEA leads to an enriched interpretation of classic inventory theory, and further development.

2.1. Equivalence framework

A firm is involved in some activity A. The activities in inventory theory are typically about the movement, transformation, and stockage of products or services. A firm has to deal with flows of goods and services inside its boundaries as well as those exchanged with the outside world. The latter may include suppliers, third parties, employees of the firm, providers of equipment and other supplies, and customers. Let X denote a scenario by which all these flows of goods and services, needed to perform A in the time interval $[0,\infty)$, are organised in a certain manner. Call $\mathcal{A}$ the set of all such scenarios available to the firm.

Using classic inventory theory, one constructs an average profit function $P(X)$ for the firm related to A that reflects both real expenses and revenues, and *capital costs* as a function of X. The problem calls for finding $X^* = \arg\max_{X\in\mathcal{A}} P(X)$. For A to be worth while, $P(X^*) > 0$. For constant revenues, one can alternatively minimise a cost function $C(X)$. In the NPV framework, one starts from the cash-flows related to A that the firm exchanges with the outside world. The cash-flow function of interest, $a(X)$, describes the size and timing of these in- and outgoing cash-flows as a function of X. The problem calls for finding $X^* = \arg\max_{X\in\mathcal{A}} \int_0^\infty a(X)e^{-\alpha t}\,dt$. The NPV (or AS) related to X^* should be non-negative to justify engagement in A.

The question of equivalence deals with the problem of establishing whether, and if so, under which conditions, the two frameworks can find the same scenario X^* to be optimal. It is now well-known that it is impossible to achieve in but trivial and non-interesting models.[1] Better results are known to be obtainable when comparing with a linear approximation of the NPV (AS) function. Notwithstanding the worth of unapproximated or higher order NPV models, see e.g. Disney and Warburton (2012), the study of linear approximations remains useful for the following reasons. First, these models often offer analytical solutions and insight. Second, they are typically accurate if we account the fact that managers commonly do not wish to implement solutions with very long cycle times. Third, at least since Hadley and Whitin (1963) classic inventory theory is thought to represent the linear approximation of NPV fairly accurately. Within the range of practical cycle times discussed above, it should be an accurate theory *if* it is indeed a theory about the linear approximation. NPVEA shows that this is in general not the case: the linear approximation is needed in proving which first order effects are responsible for the difference. This opens the route to improving the theory.

We adopt from the literature the notion of NPV models being more accurate in reference to a particular 'real-world' situation. Because a classic model is based on less information, it might be (very close to being) the linear approximation of potentially many NPV models. Equivalence to a reference model reveals applicability of the classic model to *that* situation, and the conditions under which equivalence holds help to complete this interpretation by specifying how to set the classic model's parameters. There are many possible reference models and hence possible interpretations.

It is formalised as follows. Let q be a propositional variable about the *applicability* of a particular classic inventory model. We use q to express in shorthand the belief that this model can be used. Let p be a propositional variable about the *validity* of a particular cash-flow function $a(X)$. Hence, p expresses in shorthand which NPV model is the reference. Two types of results are obtainable:

1. $p \to \neg q$, and q, infer $\neg p$ by modus ponens;
2. $p \to q$.

A result of the type $p \to \neg q$, which we call 'non-equivalence', implies the following:

- the applicability of the classic model cannot hold under p;
- an interpretation of the classic model is not obtainable from p;

[1] This follows from the fact that linear models cannot account for the cumulative impact of interests on interests, see the discussion in Haneveld and Teunter (1998).