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An improved approximation for the renewal function and its integral with an application in two-echelon inventory management

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ABSTRACT

A simple but effective approximation is proposed to compute the renewal function, $M(t)$, and its integral. The asymptotic approximation of the renewal function and its integral, which are widely used in decision makings involving a renewal process, may not perform well when t is not large enough. To overcome the inaccuracy of the asymptotic approximation, we propose a modified approximation that computes the renewal function and its integral based on the probability distribution function of inter-renewal time when the distribution function is known or based on its mean and standard deviation when the distribution function is unknown. The proposed approximation provides closed form expressions, which are important in decision makings, for the renewal function and its integral for the entire range of t rather than numerically computes them for given values of t. Extensive numerical experiments on commonly used distributions are conducted and demonstrate better performance of the proposed approximations compared to the asymptotic approximation. The new approximations are further applied to a case study of a two-echelon inventory system and result in better solutions compared to the reported results based on the asymptotic approximation.

1. Introduction and literature review

Let the random variable X_n denote the intercurrence or inter-renewal time between the $(n-1)$th and nth events in a renewal process. Assume $X_1, X_2, \ldots$ to be a sequence of nonnegative, independent random variables having a common probability distribution $F(x) = P\{X_k \le x\},\ x \ge 0,\quad k = 1, 2, \ldots,$ and $\mu_1 = E(X_i), 0 < \mu_1 < \infty$. Define $S_0 = 0,\ \ S_n = \sum_{i=1}^{n} X_i,\ \ n = 1, 2, \ldots,$ so that S_n would be the time epoch at which the nth event occurs. For each $t \ge 0$, $N(t)$ is the largest integer $n \ge 0$ so that $S_n \le t$. The random variable of $N(t)$ represents the number of events up to time t and the renewal function $M(t)$ is defined by Tijms (2003) as

$$M(t) = E[N(t)], \quad t \ge 0.$$

Define the cumulative distribution function (c.d.f.) $F_n(t) = P\{S_n \le t\}, t \ge 0, n = 1, 2, \ldots,$ and $F_1(t) = F(t)$. It is implied that $P\{N(t) \ge n\} = F_n(t),\ n = 1, 2, \ldots$. Given an $F(t)$, the renewal function $M(t)$ satisfies the integral equation of

$$M(t) = F(t) + \int_0^t M(t-x)dF(x).$$

The integral equation has a unique solution of $M(t)$, which is bounded on finite intervals under the assumption that $F(t)$ is continuous in t, $F(0) = 0$, and $F(\infty) = 1$ (Cox, 1962). Let $f(t)$ denote the corresponding probability density function (p.d.f.), if exists, for $F(t)$. Then,

$$M(t) = F(t) + \int_0^t M(t-x)f(x)dx.$$

The renewal function $M(t)$ and its integral $I(t) = \int_0^t M(x)dx$ play an important role in decision makings involving the renewal process, such as inventory planning, supply chain planning, reliability and maintenance analysis (e.g., Bahrami et al., 2000; Barlow and Proschan, 1965; Sheikh and Younas, 1985; Tijms, 1994). However, obtaining the renewal function, $M(t)$, analytically is complicated and even impossible for most distribution functions. As an analytical method, the Laplace transform $M(s)$ of the renewal function satisfies

$$M(s) = \frac{f_0(s)}{s(1-f_0(s))},$$

where $f_0(s)$ is the Laplace transforms of the density function of the inter-renewal time, $f(t)$ (From, 2001). It is usually difficult to obtain $M(t)$ through the inversion of $M(s)$ (Jaquette, 1972). We can obtain an exact computation of the renewal function, $M(t)$ for all $t \ge 0$ analytically only for a few special cases of $F(t)$ (Tijms, 2003), such as the exponential distribution. Furthermore, in many real-life

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applications the distribution for the inter-renewal time may not be known. Therefore, approximations of the renewal function have drawn much interest in the literature and result in various methods.

The asymptotic expansion is very helpful in the approximation of the renewal function and its integral because of its simplicity (Spearman, 1989). The asymptotic approximation only requires the first several moments (the first two moments for the renewal function approximation and the first three for its integral) and does not need the exact distribution function for the inter-renewal times. Because it provides a closed-form, the asymptotic approximation has been widely applied to the optimization problems that involve the renewal process, such as inventory planning, reliability and maintenance planning (e.g., Cetinkaya et al., 2008). However, asymptotic expansions for $M(t)$ are not accurate for small values of t and may yield poor optimal solutions. This paper tries to address this drawback of asymptotic approximations by proposing a new approach of approximation. At the same time, our proposed approximations keep the positive features of asymptotic approximations such as simplicity, closed-form expression for optimization, and independence from the distributions of inter-renewal times.

It is rather easy to compute $M(t)$ numerically for a given value of t (Jaquette, 1972) and a variety of approaches have been developed in the literature, such as cubic-splining algorithm by McConalogue and Pacheco (1981) to compute the renewal function by numerical convolution, the generating function algorithm by Giblin (1984), and power series expansion. The power series method is used for Weibull distribution in most studies (e.g., Wang and Pham, 1999; Weiss, 1981) but can be extended to all distributions with a power series expansion (Smeitink and Dekker, 1990). Smith and Leadbetter (1963) found an iterative solution for the case in which the inter-renewal time follows a Weibull distribution. Another iterative solution method with the Weibull distributed inter-renewal time was given by White (1964). A numerical integration approach, which covers Weibull, Gamma, Lognormal, truncated Normal and inverse Normal distributions, was offered by Baxter et al. (1982). Garg and Kalagnanam (1998) proposed a Pade approximation approach, a class of rational polynomial approximants (Baker and Graves-Morris ,1996), to solve the renewal equation for the inverse Normal distribution. Their method uses Pade approximants to compute the renewal function near the origin and switches to the asymptotic values farther from the origin. They presented a polynomial switch-over function in terms of the coefficient of variation of the distribution, enabling one to determine *a priori* if the asymptotic value can be used instead of computing the Pade approximant. A shortcoming of their method is that it does not provide a compact closed-form until applying the numerical method of Xie (1989). Kaminskiy and Krivtsov (1997) used a Monte Carlo simulation, which provides a universal numerical solution to the renewal function equation, covering essentially any parametric or empirical distribution used to model time-to-failure distributions. A method called the RS-method was established by Xie (1989) for solving renewal-type integral equations based on direct numerical Riemann–Stieltjes integration. The RS-method is particularly useful when the probability density function has singularities. The numerical method of Xie (1989) was used as a starting point of a numerical approximation proposed by From (2001) that constructs a two-piece modified rational function with the second piece being a linear function of t. An approximation for the renewal function of a failure distribution with an increasing failure rate was proposed by Jiang (2010). Although all the methods that numerically compute the renewal function are generally accurate for the small values of t but do not provide a closed-form expression that is useful for decision makings. Our proposed approximation not only is accurate in the range of small values of t but also provides a closed-form expression to facilitate optimization.

Another approach to compute the renewal function is the approximation based on $F(t)$ of a given distribution by the well known equation (Cox, 1962):

$$M(t) = \sum_{0}^{\infty} F_n(t),$$

where $F_n(t)$ is the c.d.f. of S_n, the epoch of the nth renewal and is the convolution of $f(t)$ and $F_{n-1}(t)$. For most of the distributions, it is difficult to calculate $F_n(t)$. Therefore, Gamma distribution, whose $F_n(t)$ are easy to obtain, is often used to approximate $F_2(t), F_3(t), \ldots$ based on the first two moments. The idea of exact computation of the first few terms of the renewal function in the series and approximation of the other terms using a two-moment match was developed by Smeitink and Dekker (1990). Their suggested approximation for $M(t)$ is in the form of $F(t)+\sum_{n=2}^{\infty}\overline{F}_n(t), \quad t>0$, where $\overline{F}_n(t)$ is the distribution function of $\overline{X}_1+\cdots+\overline{X}_n$ and $\overline{X}_1,\ldots,\overline{X}_n$ are independent and have a common Gamma(α,λ) distribution. The values of α and λ parameters are determined such that the first two moments of the original inter-renewal times X_i are matched by the first two moments of the Gamma(α,λ) distribution (Tijms, 2003). Their numerical experiments show that their approximation yields quick and useful approximation of the renewal function provided that coefficient of variation is not too large. However, Gamma's distribution function is already too complicated for optimization in addition to possibly complicated $F(t)$. Our proposed approximation is independent of the inter-renewal time distribution and easy to apply for decision makings.

A very important issue that has been only discussed in a small number of studies (e.g., Baxter et al., 1982) is the computation of the renewal function integral. The integral of a renewal function is extensively used in the studies that deal with the waiting and/or accumulating counting process such as inventory holding cost in inventory planning problems or cumulative damage process in reliability and maintenance problems (Zacks, 2010). Baxter et al. (1982) developed a recursively defined algorithm to numerically compute the values of renewal function and its integral for a given t but their method is not useful for cases where a closed-form expression is required for optimizing an objective function. Our work provides closed-form approximations for both the renewal function and its integral.

Approximations of the renewal function are required to meet the following three requirements of simplicity, accuracy and applicability (Jiang, 2010). Simplicity requires that the approximation has a closed-form expression and can be directly used without the need of further numerical computation. The approximation should be accurate enough from an engineering perspective within the potential value range of decision variables. The applicability means that the range of t in which the approximation is accurate should be large and it is applicable for a wide range of distribution families rather than a specific distribution. Some researchers have tried to address all of these requirements (e.g., Giblin, 1984; Kaminskiy and Krivtsov, 1997; Spearman, 1989), but they could not meet all of them, missing either one or more of the requirements due to the complicated nature of the renewal function. In this paper, we propose a simple, yet accurate and applicable, approximation of the renewal functions and their integrals. The numerical results show that our approximation performs well in the entire range of t and is easy enough to get the closed-form expressions with respect to t and plug into objective functions (e.g. cost functions) to be optimized.

Knowing more information about the inter-renewal time distribution in addition to the first two moments could make the computation of the renewal function easier and/or more accurate (Gou et al., 2008; Heisig, 1998; Jin and Liao, 2009). Gou et al. (2008) assumed that the customer arrivals would follow a Poisson