Int. J. Production Economics 149 (2014) 17–27

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

The discounted (*R*,*Q*) inventory model—The Shrewd Accountant's Heuristic

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ARTICLE INFO

Article history:
Received 27 March 2012
Accepted 16 February 2013
Available online 7 March 2013

Keywords:
Discounting
Average cost
(R,Q) inventory policies
Depreciations
Interest rates

ABSTRACT

The discounted continuous-review (*R*,*Q*) inventory model with continuous and stochastic demand is investigated. New optimality conditions are derived, clarifying the difference to the average-cost case, also graphically. Supported by depreciation theory, applied to the value of a setup, the results suggest an insightful and very precise approximation – The Shrewd Accountant's Heuristic – based on a new average-cost model. It deepens and extends the work of Hadley (1964). Three examples are worked out in detail and the model is generalized to Poisson demand and to stochastic lead-times.

1. Introduction (cf. Section 11)

We extend understanding of discounting in inventory theory by analysis of a simple version of the (*R*,*Q*) inventory model with demand following an idealized stochastic process with continuous sample path. An (*R*,*Q*) policy then coincides with an (s,S) policy and the inventory position is raised to R+Q each time it hits R. New optimality conditions are found, generalizing those of the average-cost model. The difference to the average-cost model is clarified and depicted, and a very precise heuristic, based on a new average-cost formulation and the new optimality conditions, is derived by use of depreciation theory. It is little better than the optimal average-cost policy – as is well-known – but the exercise renders insight.

Discounting is the economists' well-justified way of valuing dynamic streams of payments. It has been used for valuation of inventory policies since the beginning of inventory theory, e.g. Arrow et al. (1951), Veinott and Wagner (1965). Discounted models are frequently used to demonstrate the optimality of inventory policies, e.g., Clark and Scarf (1960), Scarf (1960). However, in practice discounted-cost inventory models are most often approximated by the more convenient average-cost formulations that charge interest cost of inventory and backlog. Quite surprisingly, comparisons of the two approaches are rare in the literature.

Hadley (1964) presented an average-cost approximation of the discounted, deterministic EOQ model. It appears in some textbooks, e.g., Zipkin (2000, ch. 3.7) and Porteus (2002, App. C) and it has been extended and commented in quite a few articles, e.g., Grubbström (1980), Grubbström and Thorstenson (1986), Haneveld and Teunter (1998), Corbacioglu and van der Laan (2007) and Beullens and Janssens (2011). Chao (1992) developed an exact model with Wiener demand.

Hadley approximated the discounted annuity by the average cost plus interest on half the setup cost. By applying depreciation theory and the new average-cost model, we improve his estimate. We then use the new cost estimate to derive a heuristic policy through the new optimality conditions. The precision of this simple heuristic turns out to be extremely high.

The sequel is organized as follows: Section 2 finds the net present value of an (*R*,*Q*) policy and the discount rate, *r* (an insight); Section 3 presents the new optimality result (Proposition 2 – a major insight). Section 4 applies depreciation theory to calculate the annuity through a (re-)discovered accounting identity (Proposition 3 – another insight) and compares the two approaches (Corollary 3, Fig. 2). Section 5 presents the heuristic, Sections 6–8 applies it to three models – the deterministic EOQ model without and with backlogging and to the model with Normal lead-time demand, Section 9 generalizes the optimality result to Poisson demand (Proposition 4) – this model is close to Johansson and Thorstenson (1996). Section 10 compares the new and the traditional average-cost formulation (an insight!) and discusses stochastic lead-times. Section 11 concludes. Two appendices contain omitted proofs and a third, omitted formulae.

2. The model

The single-item inventory model has replenishment lead-time, *L*, and expected demand rate, *D*. The demand process is infini-

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0925-5273/$ - see front matter
http://dx.doi.org/10.1016/j.ijpe.2013.02.010

tely divisible with a continuous sample path and independent, non-negative increments. Only the deterministic process fulfills these requirements, strictly speaking, but it is a common approximation in inventory theory that may lead to strikingly intuitive heuristics. The lead-time demand distribution is denoted by $F_L(\cdot)$, with density $f_L(\cdot)$ and mean $D\cdot L$. The sales price to customers is π per unit and there is a proportional procurement cost of c per unit and a fixed one, K per procurement. All unsatisfied demand is backlogged. $\tilde{G}_0(y)$ is the expected out-of-pocket cost rate (per time period) L periods ahead for holding inventory and backlog when y is the present inventory position (i.e., stock on hand plus on order minus backlogs). The generalization to stochastic lead-times is discussed in Section 10.

Discounting is continuous with an interest rate, $r>0$, made applicable to all payments. The discounting technique is nonstandard, though, because the discount function is expressed in terms of time passed for x units to be demanded. Let t_x denote this (stochastic) time. We have $E[t_x]=x/D$ by independent increments. Let $\alpha>0$ denote the regular interest rate per unit time. Then $r>0$ is defined so that $e^{-rx/D}=E[e^{-\alpha t_x}]$. Because the demand process is infinitely divisible, the value of r is independent of t_x, and because $E[t_x]=x/D$, we have that $r\le\alpha$ as the discount function is convex. In effect this change of discount function means that we measure "time" in units demanded – one "period" corresponds to D units.

$\tilde{G}_0(x)$ is a cost rate per unit real time, and to allow for discounting by r it must be adjusted: during an interval $t_{\Delta x}$ when inventory moves from x to $x-\Delta x$, the expected cost would be about $\tilde{G}_0(x)E[1-e^{-\alpha t_{\Delta x}}]/\alpha$, which approaches $r/\alpha\tilde{G}_0(x)dx$ as $x\to dx$. Thus, replacing $\tilde{G}_0(\cdot)$ by $G_0(\cdot)=r/\alpha\tilde{G}_0(\cdot)$ converts it into a cost rate per "period" that may be discounted by r.

All cost calculations are made L periods ahead, but the common discount factor $e^{-\alpha L}$ is neglected. Replenishments are paid for when received, and backlogs render revenue when delivered. Operations begin at time $-L$ by ordering $R+Q$ units. Simultaneously demand begins, so that all initial demands are backlogged until the first order arrives at time zero. Clearly the cost of this backlogging is independent of (R,Q), and we neglect it. We also neglect the expected "profit" of the initial demand, $(\pi-c)DL$, although not realized until time zero or later. In each cycle, backlog and stock left over are charged the following cycle, making the cycles identical, as each order is placed at inventory position R. The length of a cycle is Q/D "periods". The discount factor at the start of the nth cycle is $e^{-r(n-1)Q/D}$ (see further Appendix 1).

Proposition 1. *The net present value (NPV) of the (R,Q) policy over an unbounded horizon when demand follows the idealized stochastic process with continuous sample path is*

$$NPV=[(\pi-c)D-g_r]/r, \tag{1}$$

where

$$g_r=\left\{K+\int_0^{Q/D}G(R+Q-tD)\cdot e^{-rt}dt\right\}r/(1-e^{-rQ/D})$$
$$=\left\{K+\int_0^{Q}G(R+Q-x)/D\cdot e^{-r\cdot x/D}dx\right\}r/(1-e^{-rQ/D}) \tag{2}$$

and

$$G(x)=G_0(x)+rc\int_0^x(x-\xi)f_L(\xi)d\xi+r(\pi-c)\int_x^\infty(\xi-x)f_L(\xi)d\xi. \tag{3}$$

Moreover, $r\le\alpha$ = the continuous discount rate per time period.

The derivation is found in Appendix 1. Recall that $G_0(\cdot)=r/\alpha\tilde{G}_0(\cdot)$. In (2) the integral represents the expected holding and backorder costs over a cycle discounted to its beginning. Comparing with (1) one sees that g_r is computed as the net present value, g_r/r, multiplied by r. Thus, g_r should be interpreted as the expected setup, holding and backlogging cost rate per "period" corresponding to the arrival of D units of demand, i.e., it is actually a stochastic rate in regular time, proportional to realized demand; g_r/D is the similar cost per unit.

The annuity of NPV, $r\cdot NPV$, is correspondingly $(\pi-c)D-g_r$ per "period". This result is made possible by (3), i.e., we may account for the profit $(\pi-c)$ at time of demand if an interest cost $r\cdot c$ is added to the holding cost and $r(\pi-c)$ to the backlogging cost coefficient (cf. Appendix 1 and for the discrete time analog, Veinott, 1969 or Janakiraman and Muckstadt, 2004).

Multiplying the annuities by $\alpha/r\ge1$ turns them into rates per regular time period, possibly a more useful dimension when comparing with annuities of other investments, e.g. transport vehicles, but the difference seems minute, cf. Table 4.

We define a *new average-cost formulation* for the expected inventory cost rate, namely,

$$g_A=\left\{K+\int_0^{Q/D}G(R+Q-tD)dt\right\}D/Q$$
$$=\left\{K+\int_0^{Q}G(R+Q-x)/Ddx\right\}D/Q \tag{4}$$

The opportunity interest costs of holding stock and backlog are included in $G(x)$ by (3), as is typically done in practical applications. However, the *traditional average-cost formulation* is hardly based on $G(x)$ but rather on the (typically very close) $\tilde{G}(x)$ – a rate per real period:

$$\tilde{G}(x)=\tilde{G}_0(x)+\alpha c\int_0^x(x-\xi)f_L(\xi)d\xi$$
$$+\alpha(\pi-c)\int_x^\infty(\xi-x)f_L(\xi)d\xi=\alpha/rG(x) \tag{5}$$

The traditional average-cost formulation will be discussed in Section 10.

3. Main results

Proposition 2. *Let $S=R+Q$.*
Then $\partial g_r/\partial R=0=\partial g_r/\partial S$ if and only if (R^*, Q^*) satisfies

$$G(R^*)=g_r^*\text{ and }G(R^*+Q^*)=g_r^*-rK. \tag{6}$$

Proof. With $S=R+Q$,

$$g_r(R,S-R)=g_r=\left\{(K+\int_R^S G(x)e^{-r(S-x)/D}dx/D\right\}r/(1-e^{-r(S-R)/D})\quad\text{by (1)}$$

and so,

$$\partial g_r/\partial R=\left\{-(G(R)/D)e^{-r(S-R)/D}+(g_r/D)e^{-r(S-R)/D}\right\}\cdot r/(1-e^{-r(S-R)/D})$$
$$=\left\{-G(R)+g_r\right\}/D\cdot e^{-r(S-R)/D}\cdot r/(1-e^{-r(S-R)/D}). \tag{7}$$

Thus,
$\partial g_r/\partial R=0\leftrightarrow G(R^*)=g_r^*$. Next,

$$\partial g_r/\partial S=\left\{G(S)/D-(r/D)\int_R^S G(x)e^{-r(S-x)/D}dx/D-(g_r/D)e^{-r(S-R)/D}\right\}\cdot r/(1-e^{-r(S-R)/D})$$
$$=[G(S)-r\left\{g_r(1-e^{-r(S-R)/D})/r-K\right\}-g_re^{-r(S-R)/D}]/D\cdot r/(1-e^{-r(S-R)/D})$$
$$=(G(S)-g_r+rK)/D\cdot r/(1-e^{-r(S-R)/D}) \tag{8}$$

Thus, also
$\partial g_r/\partial S=0\leftrightarrow G(S^*)=g_r^*-rK$. □

For the average-cost case, with (4) interpreted as $r=0$, Proposition 2 degenerates into the well-known

$$G(R_A)=g_A^*=G(R_A+Q_A), \tag{9}$$