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Risk measurement in the presence of background risk

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Abstract

A distortion-type risk measure is constructed, which evaluates the risk of any uncertain position in the context of a portfolio that contains that position and a fixed background risk. The risk measure can also be used to assess the performance of individual risks within a portfolio, allowing for the portfolio's re-balancing, an area where standard capital allocation methods fail. It is shown that the properties of the risk measure depart from those of coherent distortion measures. In particular, it is shown that the presence of background risk makes risk measurement sensitive to the scale and aggregation of risk. The case of risks following elliptical distributions is examined in more detail and precise characterisations of the risk measure's aggregation properties are obtained.

Keywords: Risk measures; Background risk; Capital allocation; Portfolio management; Elliptical distributions

1. Introduction

Risk measures are used in order to determine the *risk capital* that the holder of a portfolio of assets and liabilities has to hold, that is, to invest with low risk. Risk measures are closely related to insurance premium calculation principles, which are studied extensively by Goovaerts et al. (1984). Recent approaches to risk measures include the one by Artzner et al. (1999), who introduced a set of axioms on risk measures and introduced the term *coherent measures of risk* for risk measures that satisfy those axioms. Risk measures which satisfy these axioms have been introduced in an insurance pricing context by Denneberg (1990) and Wang (1996), and in a finance setting by Acerbi (2002). These are often called *distortion risk measures* and are calculated as the expected loss of a portfolio, under a nonlinear transformation of its cumulative probability distribution.

Given the aggregate risk capital of a portfolio, a separate problem that emerges is the allocation of the total amount of risk capital to the instruments (or sub-portfolios) that the portfolio consists of. The term capital allocation does generally not imply that actual amounts of capital are moved between portfolios; it rather signifies a notional exercise used primarily to evaluate the performance of risks within a portfolio. Capital allocation methods drawing on cooperative game theory were proposed by Denault (2001), based on the premise that no disincentives for the pooling of portfolios should be created by the allocation.

When a distortion risk measure is used for determining the aggregate capital corresponding to the portfolio, explicit capital allocation formulas have been obtained by Tsanakas and Barnett (2003) and Tsanakas (2004). The capital allocation mechanism obtained in the above papers can be viewed as an internal system of prices defined on the set of risks that form the aggregate portfolio. The capital allocated to a particular risk, or its price, is influenced on the dependence between that risk and the aggregate portfolio.

If the aggregate portfolio is fixed, this approach presents a consistent method for the allocation of risk capital. However, if the portfolio is not fixed, and capital allocation is viewed as a decision tool for its re-balancing, then an inconsistency arises. Allocated capital reflects the dependence between a risk and the portfolio which contains that risk. Hence when the weight of that risk in the portfolio changes, the portfolio as a whole is affected. Therefore, the system of internal prices derived for the static portfolio changes and is no longer valid for assessing the performance of each individual risk in the re-balanced portfolio.

A related situation arises when the portfolio holder is also exposed to an element of *background risk*. This means that besides the specific portfolio obtained in the financial and

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insurance markets, the holder is also exposed to a risk that he cannot (or will not) trade, control or mitigate. Examples of background risk are the risk to human capital for a firm, unhedgeable portfolios in an incomplete financial market and insurance risk for which no reinsurance is typically available e.g. infinite layers. Hence, the risk measurement of any other risk in the portfolio has always to be evaluated with reference to the background risk that the holder is exposed to.

While the effect of background risk on risk measurement is not well developed in the literature, its effect on risk aversion and asset allocation has been extensively studied in the economics literature. For example, Heaton and Lucas (2000) deal with portfolio choice in the presence of background risk, while Gollier and Pratt (1996) seek to characterise utility functions such that the exposure to an independent background risk increases risk aversion with respect to any other independent risk. In this paper a different approach is taken. It is proposed that the risk corresponding to a particular position or instrument is quantified via its risk contribution to a portfolio that contains itself and the background risk. That risk contribution can be determined using standard capital allocation methods. The risk contribution's sensitivity to the exposure to background risk can then be used to characterise the change in preferences that background risk induces. No assumptions on independence of risks are made.

Following that reasoning, a new distortion risk measure is defined, which provides a measure of risk for any position X, with reference to a fixed risk Y that is in the same portfolio as X. This can be interpreted either as capturing the dynamics of capital allocation under changing portfolio weights or as quantifying risk X in the presence of a background risk Y. This is akin to the portfolio holder's having a system of reference where the origin has been moved from 0 to Y. This 'new origin' is now of course a random quantity and the risk measure of X relative to Y proposed here reflects this change.

By studying the properties of that risk measure, it is possible to capture the impact of background risk on preferences, given that these preferences are captured by a distortion risk measure. It is shown that background risk increases agents' sensitivity to the aggregation of risk; while a coherent distortion risk measure is positive homogenous and subadditive, background risk potentially induces superlinearity and superadditivity. This is a notable departure from the properties of coherent risk measures, which have been criticised by some authors (e.g. Dhaene et al. (2003)) for their insensitivity to the aggregation of risk. Increased sensitivity to risk aggregation in the context of the paper makes practical and intuitive sense; in an illiquid environment, such as an insurance market, a risk taker will generally require a higher price at the margin for risk that he is already highly exposed to.

In the case of general probability distributions and dependence structures, the aggregation properties of risk measurement in the presence of background risk are difficult to characterise exactly. Therefore, the special case of joint-elliptically distributed risks is examined. Elliptical distributions have become an important tool in financial risk management, because they combine heavy tails and tail-dependence with tractable aggregation properties (Embrechts et al., 2002). It is shown that in the presence of background risk, the risk measure introduced here satisfies a superlinearity property which manifests sensitivity to the exposure to 'large' risks. Moreover, it is shown that the risk measure is superadditive for comonotonic risks, which means that the pooling of perfectly correlated positions is penalised. Finally it is shown that, under some conditions (ensuring that the background risk does not operate as a hedge), the risk measure penalises increases in the variability of liabilities, as well as increases in the correlation between instruments within a portfolio.

The structure of the paper is as follows. In Section 2 capital allocation with distortion risk measures is introduced. In Section 3 distortion risk measures with reference to a background risk are introduced and their basic properties discussed. Section 4 discusses elliptical distributions and proceeds with the characterisation of the risk measure's aggregation properties in an elliptical environment. Conclusions are summarised in Section 5.

2. Capital allocation with distortion risk measures

Consider a set of random variables $\mathcal{X}$ representing insurance and financial risks, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. When $X \in \mathcal{X}$ assumes a positive value it is considered a loss. A *risk measure* ρ is defined as a real-valued functional on the set of risks, $\rho : \mathcal{X} \mapsto \mathbb{R}$. The quantity $\rho(X)$ represents the amount of capital that the holder of risk X has to safely invest, in order to satisfy a regulator. Risk measures have been extensively studied in actuarial science in the guise of insurance pricing functionals; see Goovaerts et al. (1984). A more recent influential approach in the financial field has been the axiomatic definition of 'coherent measures of risk' by Artzner et al. (1999).

The following *distortion risk measure* has been proposed by Denneberg (1990) and Wang (1996):

$$\rho(X) = \int_{-\infty}^{0} \left(g\left(\bar{F}_X(x)\right) - 1\right) \mathrm{d}x + \int_{0}^{\infty} g\left(\bar{F}_X(x)\right) \mathrm{d}x, \qquad (1)$$

where $\bar{F}_X(x) = \mathbb{P}(X > x)$ is the decumulative probability distribution of X under the real-world probability measure $\mathbb{P}$, while $g : [0,1] \mapsto [0,1]$ is an increasing and concave *distortion function* such that $g(0) = 0$ and $g(1) = 1$. The distortion risk measure can be interpreted as an expected loss under a nonlinear transformation of the probability distribution; observe that if $g(t) = t$, then $\rho(X) = E[X]$. The distortion risk measure (1) satisfies the coherence axioms of Artzner et al. (1999), while allowing an economic interpretation as a certainty equivalent under Yaari (1987) dual theory of choice under risk. It is noted that distortion risk measures are essentially the same as the 'spectral risk measures' of Acerbi (2002).

Consider the case where g is differentiable on $[0,1]$ with bounded first derivative, an assumption that is made throughout the paper. Moreover, we assume that $\bar{F}_X$ is continuous. Then integration by parts yields the following re-writing of the distortion risk measure:

$$\rho(X) = E\left[X g'\left(\bar{F}_X(X)\right)\right]. \qquad (2)$$