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A note on the Markov property of stochastic processes described by nonlinear Fokker–Planck equations

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Abstract

We study the Markov property of processes described by generalized Fokker–Planck equations that are nonlinear with respect to probability densities such as mean field Fokker–Planck equations and Fokker–Planck equations related to generalized thermostatistics. We show that their transient solutions describe non-Markov processes. In contrast, stationary solutions can describe Markov processes. As a result, nonlinear Fokker–Planck equations can be used to model transient non-Markov processes that converge to stationary Markov processes.

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In general, stochastic processes can be characterized by means of transition probability densities. In the trivial case, transition probability densities depend on a single time-point and we deal with pure random processes. In the simplest, nontrivial case transition probability densities depend on two time-points. Then, we deal with Markov processes [1]. In view of this property, Markov processes play an important role in the theory of stochastic processes. Moreover, it has been found that many stochastic processes observed in physics and other sciences can indeed be regarded as Markov processes [2–5]. However, Markov processes describe an idealized situation [6]. In

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the general case, there is an effect of the history of a system on its current behavior. Such long-term memory effects cannot be taken into account by Markov processes and require a description in terms of non-Markov processes. Therefore, to differentiate between Markov and non-Markov processes basically means to discuss the relevance of long-term memories of systems. Such a discussion can be carried out by means of appropriately defined stochastic models. To this end, however, we need to know whether or not the models describe Markov processes.

Recently, there has been an increasing interest in modeling stochastic processes by means of Fokker–Planck equations that are nonlinear with respect to their probability densities. Such processes have been used, for example, to describe synchronization phenomena [3,7–12], muscular contractions [13,14], noise-induced phase transitions [15,16], and nonextensive systems [17–24] (see also Ref. [25]). In spite of an increasing number of applications, little attention has been directed towards the Markov property of processes described by nonlinear Fokker–Planck equations. In fact, in literature one can find brief comments on this issue. However, these comments are controversial [26–36]. Some authors have suggested that nonlinear Fokker–Planck equations describe Markov processes, others have mentioned that they describe non-Markov processes. In addition, some authors prefer to avoid using the terms Markov or non-Markov processes and have called stochastic processes described by nonlinear Fokker–Planck equations "nonlinear Markov processes".

In this context, frequently, the claim has been made that the stochastic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = h(\xi) + \langle g(\xi - x)\rangle_{P(x,t)} + \sqrt{Q}\Gamma(t) \tag{1}$$

with $Q > 0$, $\langle g(z-x)\rangle_P = \int g(z-x)P(x,t)\,\mathrm{d}x$, $P(x,t) = \langle x - \xi(t)\rangle$, and Γ defined as Langevin force [1] corresponds to the nonlinear Fokker–Planck equation

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}[h(x) + \langle g(x-z)\rangle_{P(z,t)}]P(x,t) + \frac{Q}{2}\frac{\partial^2}{\partial x^2}P(x,t)\ . \tag{2}$$

Likewise, it has been claimed that the stochastic differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) = h(\xi) + \sqrt{QP(x,t)^{q-1}}\,\Gamma(t) \tag{3}$$

with $q > 0$ corresponds to the nonlinear Fokker–Planck equation

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}h(x)P(x,t) + \frac{Q}{2}\frac{\partial^2}{\partial x^2}P^q(x,t)\ . \tag{4}$$

In both cases, the claims have been made by referring to standard textbooks on Markov processes. That is, the authors of these studies have tacitly assumed that nonlinear Fokker–Planck equations describe Markov processes. In this note, we will show that nonlinear Fokker–Planck equations cannot describe transient Markov processes. Therefore, we cannot use the theory of Markov processes in order to show the equivalence of the stochastic differential equations (1) and (3) with the nonlinear Fokker–Planck equations (2) and (4). Nevertheless, Eqs. (1) and (3) can be derived from Eqs. (2)