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Diffusion and memory effects for stochastic processes and fractional Langevin equations

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Abstract

We consider the diffusion processes defined by stochastic differential equations when the noise is correlated. A functional method based on the Dyson expansion for the evolution operator, associated to the stochastic continuity equation, is proposed to obtain the Fokker–Planck equation, after averaging over the stochastic process. In the white noise limit the standard result, corresponding to the Stratonovich interpretation of the non-linear Langevin equation, is recovered. When the noise is correlated the averaged operator series cannot be summed, unless a family of time-dependent operators commutes. In the case of a linear equation, the constraints are easily worked out. The process defined by a linear Langevin equation with additive noise is Gaussian and the probability density function of its fluctuating component satisfies a Fokker–Planck equation with a time-dependent diffusion coefficient. The same result holds for a linear Langevin equation with a fractional time derivative (defined according to Caputo, Elasticità e Dissipazione, Zanichelli, Bologna, 1969). In the generic linear or non-linear case approximate equations for small noise amplitude are obtained. For small correlation time the evolution equations further simplify in agreement with some previous alternative derivations. The results are illustrated by the linear oscillator with coloured noise and the fractional Wiener process, where the numerical simulation for the probability density and its moments is compared with the analytical solution.

Keywords: Stochastic processes; Correlated noise; Functional methods; Fractional derivatives

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0. Introduction

Stochastic equations are a basic tool to model physical and biological systems, since the addition of a noise term in deterministic dynamical models allows to take into account, in a phenomenological way, the coupling of neglected degrees of freedom or more generally the fluctuations of external fields, describing the environment. The simplest case to treat is the white noise, since the absence of correlations allows to simulate the system in a straightforward way and to write the parabolic Fokker–Planck equation, which governs the time evolution of probability density function (PDF). The functional method, where the Dyson expansion [1,2] for the evolution operator associated to the continuity equation is averaged, appears to be well suited to derive the Fokker–Planck equation in the case of white noise [3–6]. In any real process the noise is correlated and the white noise limit can be justified only if the decay of correlation is very rapid. If the noise has a dynamical origin (chaotic degrees of freedom) the memory effects are usually relevant and an adequate treatment is required. Memory effects have also been introduced by replacing the ordinary time derivative with a fractional derivative and the corresponding linear case will be considered.

We propose to use the Dyson series [2] to analyze the evolution of the probability density for the process defined by a Langevin equation with correlated noise. The starting point is the continuity equation, which holds provided that the Langevin equation is an ordinary differential equation, for any realization of the noise. By averaging the Dyson series over the stochastic process we obtain an operator series whose resummation, whenever possible, leads to an evolution equation for the probability density.

In the white noise limit the resummation process of the formal series expansion leads to the ordinary Fokker–Planck equation, which corresponds to the Stratonovich interpretation of the Langevin equation. When the noise is correlated an exact resummation cannot be achieved, unless a commutation condition is satisfied: in this case the PDF satisfies a generalized Fokker–Planck equation with a drift and a second-order integro-differential operator. For a linear Langevin equation with correlated noise the constraints on the coefficients imposed by the commutation condition are easily worked out. In the generic linear or non-linear case an approximate evolution equation for the PDF based on a small noise expansion is proposed. The additional expansion for small correlation time leads to a simplified equation, the same found by Risken [7] following a different procedure. The same approximation has been obtained by various authors using different methods [8–11].

The proposed method is applicable to any stochastic differential equation for which a continuity equation holds and allows to recover in a rather simple way the exact or approximate equations for the corresponding PDF, previously derived by other methods [7,12,13].

Langevin equations with delay or a memory kernel require a different approach because the state at time t depends on the previous history of the system, preventing to write the continuity equation as a first-order partial differential equation.

The only case of Langevin equations with a memory kernel, which can be treated is the linear case. Their treatment is the same which applies to any linear equation with a white or correlated noise. A stochastic process x, driven by $\eta(t)$, solution of a Langevin