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Modeling of bounded stochastic processes

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Abstract

Random processes of bounded variation are generated by using of randomized sinusoidal model and nonlinear filter model. In the randomized sinusoidal model, random noises ar\e introduced in phase angles; while in the nonlinear filter model, a set of nonlinear Itô differential equations are employed. In both methods, the spectral density of a modeled random process can be matched by adjusting model parameters. However, the probability density of the process generated by the randomized sinusoidal model has a fixed shape, and cannot be adjusted. On the other hand, the nonlinear filter model covers a variety of profiles of probability distributions.

Keywords: Random processes; Stochastic differential equations; Probability density; Spectral density

1. Introduction

When investigating a dynamical system under random excitations, it is important that each excitation process should be modeled properly to resemble its measured or estimated statistical and probabilistic properties. In many cases, Gaussian distribution is assumed for convenience of analysis. However, the range of Gaussian distribution is unbounded; namely, there exists the probability of having very large values. This violates the very nature of a real physical quantity, which is always bounded. In the reliability analysis of a physical system, the allowable failure probability is usually very small. Thus, the assumption of Gaussian distribution may affect the reliability estimation significantly.

Based on such a consideration, one versatile model for bounded random processes has been proposed [1,2] by using a sinusoidal function with a constant amplitude, a constant average frequency, and a random phase varying as a Wiener process. Such a random process is bounded by the constant amplitude assigned in the model. It is capable of modeling a random process possessing a one-peak spectrum which can be either broad band or narrow band. It has been used, for example, to investigate a straight pipe with a slug flow of a two-phase fluid [3], a structure with a spatially disordered traveling parametric excitation [4], and log-span bridges in turbulent winds [5]. However, the probability density of such a random process cannot be adjusted. Moreover, it has a singular shape with a minimum value at the zero and approaching infinity at the boundaries, which may be inappropriate in many practical cases. It is known that the shape of the probability distribution of the excitation process may have a significant effect on the response of certain types of systems, especially when the system behaviors during the transient state are of interest [6,7]. For such cases, the sinusoidal model may be not applicable.

Another type of bounded random processes were generated using nonlinear filters [8], in which the Itô type stochastic differential equations are employed with the drift coefficient adjusted to match the spectral density and the diffusion coefficient adjusted to match the probability density. Although the procedure is capable of generating both unbounded and bounded random processes, it is especially suitable to model bounded random processes with different types of probability distributions. However, the general procedure introduced in Ref. [8] is limited to one-dimensional Itô differential equations which could only model low-pass random processes, namely, processes with spectrum peaks at zero frequency. This is a severe restriction since random processes in many engineering problems have spectra with peaks at nonzero frequencies or multiple peaks.

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In this article, the randomized sinusoidal model is reviewed first. It is then extended to model a process with more than one peak in its spectrum. The probability distribution of such a process is also investigated. To extend the method of nonlinear filters to model random processes with spectrum peaks at nonzero frequencies and with multiple spectrum peaks, multi-dimensional Itô stochastic equations are constructed. The spectral densities of the processes can be adjusted by parameters in drift coefficients, and the probability distributions can be of different shapes adjusted by parameters in diffusion coefficients. In both methods, parameters can be selected to model processes with either narrow band or broad band.

2. Randomized sinusoidal model

2.1. Bounded processes with one spectrum peak

A bounded random process can be modeled as

$$X(t) = A\cos[\omega_0 t + \sigma B(t) + \theta] \tag{1}$$

where A, ω_0 and σ are positive constants, $B(t)$ is a unit Wiener process, and θ is a random variable uniformly distributed in $[0,2\pi]$. The incursion of θ renders process $X(t)$ weakly stationary. The spectral density of $X(t)$ can be found as

$$\Phi_{XX}(\omega) = \frac{A^2\sigma^2(\omega^2 + \omega_0^2 + \sigma^4/4)}{4\pi[(\omega^2 - \omega_0^2 - \sigma^4/4)^2 + \sigma^4\omega^2]} \tag{2}$$

and its probability density is

$$p_X(x) = \frac{1}{\pi\sqrt{A^2 - x^2}}, \qquad -A \le x \le A \tag{3}$$

Fig. 1 shows the spectral densities in the positive ω range for the case of $\omega_0 = 3$ and several values of noise level σ. It is seen that the spectral densities reach their peaks near

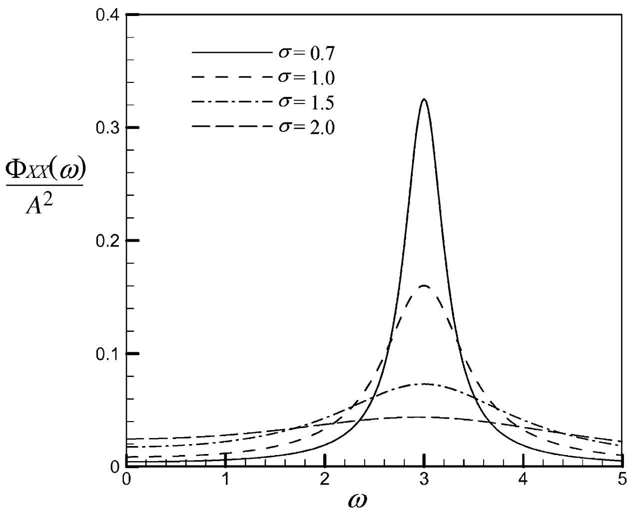

Fig. 1. Spectral density of $X(t)$ generated from randomized sinusoidal model (1) with $\omega_0 = 3$.

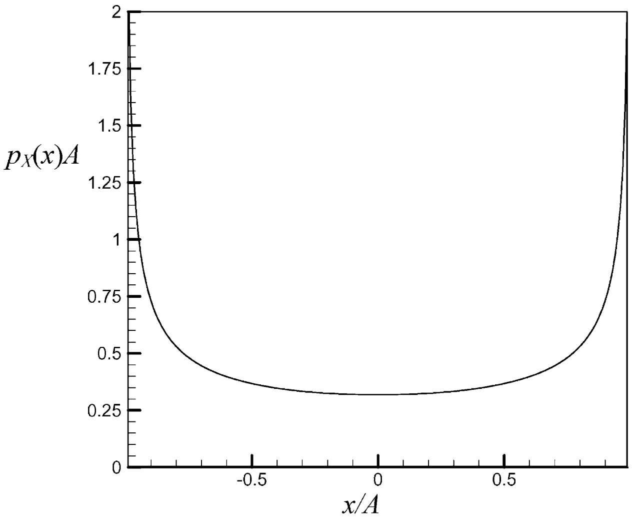

Fig. 2. Probability density of $X(t)$ generated from randomized sinusoidal model (1).

$\omega = \omega_0$, and possess different band widths for different noise levels. Fig. 2 depicts the probability density function for the processes. It has very large values near the two boundaries. Note that the spectral density can be adjusted in terms of parameters A, ω_0 and σ; while the probability density only depends on A. In a practical problem, the boundary A is determined according to the physical phenomenon; thus, the probability distribution is not adjustable.

2.2. Bounded processes with multiple spectrum peaks

The randomized sinusoidal model can be extended to include more terms, as given by

$$X(t) = \sum_{i=1}^{n} A_i\cos[\omega_i t + \sigma_i B_i(t) + \theta_i] \tag{4}$$

where A_i are positive constants, $B_i(t)$ are mutually independent unit Wiener processes, and θ_i are mutually independent random variables uniformly distributed in $[0,2\pi]$. The spectral density of $X(t)$ is now

$$\Phi_{XX}(\omega) = \sum_{i=1}^{n} \frac{A_i^2\sigma_i^2(\omega^2 + \omega_i^2 + \sigma_i^4/4)}{4\pi[(\omega^2 - \omega_i^2 - \sigma_i^4/4)^2 + \sigma_i^4\omega^2]} \tag{5}$$

and the probability density can be calculated from

$$p_X(x) = \int_D p_{Y_1}(y_1)p_{Y_2}(y_2)\ldots p_{Y_{n-1}}(y_{n-1}) \times p_{Y_n}(x - y_1 - y_2 - \cdots - y_{n-1})\mathrm{d}y_1\,\mathrm{d}y_2\cdots\mathrm{d}y_{n-1} \tag{6}$$

where

$$p_{Y_i}(y_i) = \frac{1}{\pi\sqrt{A_i^2 - y_i^2}}, \qquad i = 1,2,\ldots,n \tag{7}$$

and the integration domain D is $(n-1)$-dimensional and determined according to x and A_i. For the case of two terms