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Multistep predictions for multivariate GARCH models: Closed form solution and the value for portfolio management ☆

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A B S T R A C T

This paper derives the closed form solution for multistep predictions of the conditional means and covariances for multivariate ARMA-GARCH models. These predictions are useful e.g. in mean-variance portfolio analysis when the rebalancing frequency is lower than the data frequency. In this situation the conditional mean and the conditional covariance matrix of the cumulated higher frequency returns are required as inputs in the mean-variance portfolio problem. The empirical value of the result is evaluated by comparing the performance of quarterly and monthly rebalanced portfolios using monthly MSCI index data across a large set of GARCH models. Using correct multistep predictions generally results in lower risk and higher returns.

1. Introduction

This paper derives the closed form solution for multistep predictions of the conditional means and covariances from multivariate GARCH models. These predictions are useful in mean-variance portfolio analysis, when the rebalancing frequency is lower than the data frequency. In the application the empirical value of this result is evaluated in the performance of quarterly rebalanced portfolios based on correct three-step predictions. We compare their performance with that of quarterly rebalanced portfolios incorrectly based on one-step predictions and with the performance of monthly rebalanced portfolios. We use monthly Morgan Stanley Capital International (MSCI) index data for six regions.

Multistep prediction in GARCH models has been considered previously in e.g. Baillie and Bollerslev (1992). They derive the minimum mean squared error forecasts for the conditional mean and the conditional variance of univariate GARCH processes. We extend their results to the multivariate case and derive closed form representations for the conditional mean and the conditional covariances h-steps ahead. In addition we derive the explicit formula for the conditional covariance of the sum of the conditional means up to h-steps ahead. This corresponds to the conditional variance of the cumulative returns over an h-period horizon, when modelling asset returns.

In our empirical application portfolios are adjusted quarterly based on GARCH models estimated with monthly data. This implies that the conditional variances of monthly returns cumulated over three months have to be computed. The empirical part of our study is related to Ledoit, Santa-Clara and Wolf (2003), who apply one-step predictions from multivariate GARCH models for

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portfolio selection using — as we do — MSCI regional indices. However, our study is based on multistep predictions and the results are based on a larger set of GARCH models.

In particular, the *value* of the derived multistep predictions for portfolio management is evaluated on monthly data for six regional MSCI indices during the evaluation period January 1992 to December 2003. The minimum variance portfolios are tracked for 48 different GARCH models, both for monthly and quarterly rebalancing. In the latter case the quarterly rebalanced portfolios *correctly* based on multistep predictions and those *incorrectly* based on one-step predictions are evaluated. We find that using correct multistep predictions generally results in lower risk and higher returns. Furthermore, the correctly computed quarterly rebalanced portfolios exhibit higher returns than monthly rebalanced portfolios.

The paper is organized as follows: In Section 2 the multistep prediction problem is discussed. Section 3 contains the empirical application in portfolio management. Section 4 briefly summarizes and provides conclusions.

2. Multistep prediction in multivariate GARCH models

This section derives the closed form solution for the multistep minimum mean squared error (MSE) prediction of the conditional means, conditional variances and conditional covariances for multivariate GARCH models. Based on these results we also present the solution for the conditional variance of the sum of the predictions over h-periods. The results of this section can be used for the prediction of *cumulative* returns and their covariance matrices in mean-variance portfolio analysis as explained in Section 3.

Since the original contribution of Engle (1982) a large variety of ARCH and GARCH models has been proposed for volatility modelling, see Bollerlsev, Engle and Nelson (1994) or Gourieroux (1997) for early discussions of some of the models developed or Bauwens, Laurent and Rombouts (2006) and Li, Ling and McAleer (2002) for more recent surveys.

We consider a multivariate ARMA process with GARCH errors to model the dynamic behavior of the (conditional) first and second moment of the returns. Let us denote with $r_t \in \mathbb{R}^n$ the vector of returns for n assets. The *mean equation* is of the form

$$r_t = c + A_1 r_{t-1} + \cdots + A_p r_{t-p} + \varepsilon_t + B_1 \varepsilon_{t-1} + \cdots + B_q \varepsilon_{t-q}, \tag{1}$$

with $A_i, B_j \in \mathbb{R}^{n\times n}$. Here ε_t is an n-dimensional random variable such that

$$\varepsilon_t = z_t \Sigma_t^{1/2}, \tag{2}$$

where z_t is i.i.d. with $\mathbb{E}(z_t) = 0$ and $\mathrm{var}(z_t) = I$. Throughout the paper I denotes the $n \times n$ identity matrix. $\sum_t \in \mathbb{R}^{n\times n}$ is a positive definite, time-dependent covariance matrix measurable with respect to the information set at time $t-1$.

If the investment horizon is larger than one period, predictions for the *cumulative* returns are needed, which in turn require *multistep* predictions. The cumulative returns over an h-period horizon, henceforth denoted as $r_{[t+1:t+h]}$, are straightforwardly calculated from the single period returns, r_{t+i}, as follows[1]

$$r_{[t+1:t+h]} = r_{t+1} + \cdots + r_{t+h}. \tag{3}$$

Thus, the conditional variance matrix of the cumulative returns $r_{[t+1:t+h]}$ is

$$var_t\left(r_{[t+1:t+h]}\right) = var_t(r_{t+1} + \cdots + r_{t+h}) = \sum_{i=1}^{h} var_t(r_{t+i}) + \sum_{i,j=1, i\neq j}^{h} cov_t\left(r_{t+i}, r_{t+j}\right), \tag{4}$$

where throughout the paper the subscript t in $\mathbb{E}_t$, var_t and cov_t indicates that the expected value, variance respectively covariance is conditional upon the information set at time t. From Eq. (4) the calculation of $\mathrm{var}_t(r_{[t+1:t+h]})$ requires the MSE predictors of r_{t+i} for $i=1,\ldots,h$ and the corresponding conditional variances and covariances. The general formula for computing the required multistep predictions of the conditional variances of r_{t+i} from multivariate ARMA (p,q)-GARCH (k,l) models is presented below. This result is a generalization of the analogue multistep prediction for *univariate* GARCH models discussed in Baillie and Bollerslev (1992).[2] Two remarks on the discussion below are in order: First, the limits for prediction horizon $h\to\infty$ of the results for the minimum MSE predictors of the mean and variance are finite only for stationary processes. Second, the derivations below do not apply to ARCH-in-Mean type models. Detailed derivations are available in our earlier working paper Hlouskova, Schmidheiny and Wagner (2004).

For the derivation of the minimum MSE predictors of r_{t+1}, $r_{[t+1:t+h]}$ and their conditional second moments it is convenient to express the ARMA mean Eq. (1) in companion form, compare e.g. Baillie (1987, p. 108):

$$\underbrace{\begin{bmatrix} r_t \\ r_{t-1} \\ \vdots \\ r_{t-p+1} \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}}_{R_t} = \underbrace{\begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{K_1 c} + \underbrace{\begin{bmatrix} A_1 & & \cdots & & A_p & B_1 & \cdots & & B_q \\ I & 0 & \cdots & & 0 & 0 & \cdots & & 0 \\ \vdots & & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ 0 & & \cdots & I & 0 & 0 & \cdots & & 0 \\ 0 & & \cdots & & 0 & 0 & \cdots & & 0 \\ 0 & & \cdots & & 0 & I & \cdots & & 0 \\ \vdots & & \cdots & & \vdots & \vdots & \cdots & & \vdots \\ 0 & & \cdots & & 0 & 0 & \cdots & I & 0 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} r_{t-1} \\ r_{t-2} \\ \vdots \\ r_{t-p} \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}}_{R_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{K\varepsilon_t} \tag{5}$$

[1] This follows directly from the definition of the one-period returns, calculated as the logarithmic difference of asset prices.

[2] Alternatively, the temporal aggregation results of Drost and Nijman (1993), derived for a specific class of univariate GARCH models, can be used to obtain multistep predictions.