Insurance: Mathematics and Economics 50 (2012) 94–98

Contents lists available at SciVerse ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Translation-invariant and positive-homogeneous risk measures and optimal portfolio management in the presence of a riskless component

Zinoviy Landsman *, Udi Makov

Department of Statistics, University of Haifa, Israel

ARTICLE INFO

Article history:
Received April 2011
Received in revised form
July 2011
Accepted 13 August 2011

Keywords:
Riskless component
Translation-invariant and positive-homogeneous risk measure
Value-at-risk
Tail condition expectation
Minimization of root of quadratic functional
Elliptical family

ABSTRACT

Risk portfolio optimization, with translation-invariant and positive-homogeneous risk measures, leads to the problem of minimizing a combination of a linear functional and a square root of a quadratic functional for the case of elliptical multivariate underlying distributions.

This problem was recently treated by the authors for the case when the portfolio does not contain a riskless component. When it does, however, the initial covariance matrix Σ becomes singular and the problem becomes more complicated. In the paper we focus on this case and provide an explicit closed-form solution of the minimization problem, and the condition under which this solution exists. The results are illustrated using data of 10 stocks from the NASDAQ Computer Index.

1. Introduction

In this paper we consider the problem of optimal portfolio selection within the class of translation-invariant and positive-homogeneous (TIPH) risk measures, popular in actuarial and financial contexts. Important members of this class are the value-at-risk (VaR) and the Tail Condition Expectation (TCE) (also known as *Tail VaR (TVaR)* or *Expected Shortfall*), which are suggested in BASEL II and SOLVENCY II. Notice that the variance premium risk measure (expected quadratic utility) associated with the classical mean variance (MV) model (see Markowitz, 1952, Boyle et al., 1998, Section 6; Steinbach, 2001, McNeil et al., 2005, Section 6.1.5) is not a member of the TIPH-class.

For risk returns following the multivariate normal model, optimal portfolio management with the TIPH-class leads to the optimization of a combination of linear and square root of quadratic functionals of portfolio weights. This phenomenon is also preserved for the more general multivariate elliptical class of risk distributions. This class is attractive for modeling heavy-tailed asset or loss returns (see Owen and Rabinovitch, 1983 and Bingham and Kiesel, 2002). Landsman and Makov (2011), who investigated the TIPH-class, provided a simple and feasible condition for the existence of the solution as well as an analytical solution to the optimization problem when none of the returns are riskless. In this paper we provide an analytical closed form expression for the optimal portfolio when one of the returns is riskless.

Let the vector of random variables $\mathbf{X}^T = (X_1, X_2, \ldots, X_n)$ of asset returns be multivariate elliptically distributed, written as $\mathbf{X} \backsim \mathbf{E}_n(\boldsymbol{\mu}, \Sigma, g)$, namely, the density function of vector $\mathbf{X}$ can be expressed

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{c_n}{\sqrt{|\Sigma|}} g_n \left[\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right], \tag{1.1}$$

for some column-vector $\boldsymbol{\mu}$, $n \times n$-positive-definite matrix $\Sigma = \|\sigma_{ij}\|_{i,j=1}^n$, and for function $g_n(t)$, called the density generator (see details in Fang et al., 1990).

A risk measure, which may be denoted by ρ, is defined to be a mapping from the space of risks (random variables) $\mathfrak{X}$ to the real line R. In effect, we have $\rho : \mathfrak{X} \ni X \to \rho(X) \in R$.

The TIPH-class of risk measures is defined as the class of risk measures that satisfies the following two conditions:

1. Translation invariance: for any constant α, one has that

   $\rho(X + \alpha) = \rho(X) + \alpha.$

2. Positive homogeneity: for any positive constant $\gamma > 0$, one has that

   $\rho(\gamma X) = \gamma \rho(X).$

* Corresponding author.
E-mail address: landsman@stat.haifa.ac.il (Z. Landsman).

0167-6687/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.insmatheco.2011.08.002

This class includes, in addition to the VaR and TCE (TVaR) risk measures, the distorted function-based risk measure Wang (1995), Wang (1996), Dhaene et al. (2006) and the Tail Standard Deviation (TSD) risk measure Furman and Landsman (2006).

Let $R = \mathbf{x}^T\mathbf{X}$ be the portfolio return, where $\mathbf{x}^T = (x_1, \ldots, x_n)$ is the vector of real numbers. Let the risk measure ρ, a member of the TIPH-class, be related to losses $L = -R$ and, therefore, is to be minimized. Then we can write

$$\rho(-R) = -\boldsymbol{\mu}^T\mathbf{x} + \rho\left(\frac{\mathbf{x}^T\mathbf{X} - \boldsymbol{\mu}^T\mathbf{x}}{\sqrt{\mathrm{var}(\mathbf{x}^T\mathbf{X})}}\sqrt{\mathrm{var}(\mathbf{x}^T\mathbf{X})}\right)$$
$$= -\boldsymbol{\mu}^T\mathbf{x} + \rho(Z)\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}, \quad (1.2)$$

where $Z = \mathbf{x}^T(\mathbf{X} - \boldsymbol{\mu})/\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}$. The property of an elliptical family that

$$A\mathbf{X} + b \sim E_m\left(A\boldsymbol{\mu} + b, A\boldsymbol{\Sigma}A^T, g_m\right)$$

for a $m \times n$ matrix A of rank $m \le n$, implies that Z is distributed as $E_1(0, 1, g_1)$, and so the distribution of Z, a standard univariate elliptical random variable, does not depend on vectors $\boldsymbol{\mu}, \mathbf{x}$, or matrix Σ, and consequently, nor does $\rho(Z)$. In a special case, when $\mathbf{X}$ is n-dimensional normal distribution, $\mathbf{X} \sim N_n(\boldsymbol{\mu}, \Sigma)$, Z has the standard normal distribution, $N(0, 1)$.

For a value-at-risk as a TIPH risk measure, $\rho(\cdot) = \mathrm{VaR}_q(\cdot)$, (1.2) holds with

$$\rho(Z) = Z_q = \mathrm{VaR}_q(Z) = F_Z^{-1}(q). \quad (1.3)$$

If the mean vector exists, it coincides with vector $\boldsymbol{\mu}$, and then the tail condition expectation $TCE_q(X)$ exists and is a TIPH risk measure. For details see Landsman and Makov (2011), where $\rho(Z)$ is also discussed for tail standard deviation risk.

The problem of the minimization of risk measure from the TIPH class is equivalent to that of the problem of the minimization of the functional

$$\rho(-R) = -\boldsymbol{\mu}^T\mathbf{x} + \lambda\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}, \quad \lambda > 0, \quad (1.4)$$

where $\boldsymbol{\mu} = E\mathbf{X}$ and $\Sigma = \mathrm{cov}(\mathbf{X})$. This is a combination of the linear functional and square root of a quadratic functional with a balance parameter $\lambda > 0$, subject to the linear constraint

$$\mathbf{1}^T\mathbf{x} = 1, \quad (1.5)$$

where $\mathbf{1}$ is the vector-column of n ones. However, if one of the asset returns is riskless, Σ is singular and the considered optimization problem requires a new methodology which is the central topic of this paper.

Section 2 is devoted to the solution of the optimization problem when the model contains the riskless component. In Section 3 we illustrate the results using the data of 10 stocks from the NASDAQ Computer Index and the return of the riskless component.

2. Analytic solution of optimization problem with riskless component

In this section we consider the problem of minimization of functional

$$\mathbf{a}^T\mathbf{x} + \lambda\sqrt{\mathbf{x}^T\Sigma\mathbf{x}}$$

subject to a system of linear restrictions presented in the matrix form

$$B\mathbf{x} = \mathbf{c}, \quad \mathbf{c} \ne \mathbf{0}, \quad (2.6)$$

$B = (b_{ij})_{i,j=1}^{m,n}$ is $m \times n$, $m < n$, rectangular matrix of full rank, $\mathbf{c}$ is some $m \times 1$ vector and $\mathbf{0}$ is a vector-column of m zeros. Of course choosing $\mathbf{a} = -\boldsymbol{\mu}$ and $B = \mathbf{1}^T$ we reduce our problem to the one discussed in (1.4) and (1.5).

Choosing the first $n-m$ variables we have the natural partition of vector $\mathbf{x}^T = (\mathbf{x}_1^T, \mathbf{x}_2^T)$, $\mathbf{x}_1 = (x_1, \ldots, x_{n-m})^T$, $\mathbf{x}_2 = (x_{n-m+1}, \ldots, x_n)^T$ and the corresponding partition of vectors $\boldsymbol{\mu}^T = (\boldsymbol{\mu}_1^T, \boldsymbol{\mu}_2^T)$, $\mathbf{1}^T = (\mathbf{1}_1^T, \mathbf{1}_2^T)$, matrix Σ,

$$\Sigma = \begin{pmatrix} \Sigma_{\mathbf{11}} & \Sigma_{\mathbf{12}} \\ \Sigma_{\mathbf{21}} & \Sigma_{\mathbf{22}} \end{pmatrix} \quad (2.7)$$

and matrix

$$B = \begin{pmatrix} B_{\mathbf{21}} & B_{\mathbf{22}} \end{pmatrix},$$

wherematrices $B_{\mathbf{21}}$ and $B_{\mathbf{22}}$ are of dimensions $m \times (m-n)$ and $m \times m$, respectively. As matrix B is of full rank, we suppose without loss of generality that matrix $B_{\mathbf{22}}$ is non singular.

Suppose now that one of the components, say X_n, of the portfolio of returns is riskless. This means that $\tilde{\mathbf{X}} = (X_1, \ldots, X_{n-1})^T \sim E_n(\tilde{\boldsymbol{\mu}}, \tilde{\Sigma}, g_{n-1})$, where $\tilde{\boldsymbol{\mu}} = (\mu_1, \ldots, \mu_{n-1})^T$, and $\mu_n = r$, the interest rate. This implies that matrix Σ is of the form

$$\Sigma = \begin{pmatrix} \tilde{\Sigma} & \tilde{\mathbf{0}} \\ \tilde{\mathbf{0}}^T & 0 \end{pmatrix}, \quad (2.8)$$

$\tilde{\mathbf{0}}$ is vector of $(n-1)$ zeros, and consequently singular, but $\tilde{\Sigma} > 0$.

Recall $\Sigma_{\mathbf{11}} = (\sigma_{ij})_{i,j=1}^{n-m}$. For $m > 1$, define matrices $\Sigma_{\mathbf{1\tilde{2}}}$, $\Sigma_{\mathbf{\tilde{2}\tilde{2}}}$

$$\Sigma_{\mathbf{1\tilde{2}}} = (\sigma_{ij})_{i=1,,j=n-m+1}^{n-m,n-1}, \qquad \Sigma_{\mathbf{\tilde{2}\tilde{2}}} = (\sigma_{ij})_{i,j=n-m+1}^{n-1,n-1}.$$

Then matrix $\tilde{\Sigma}$ has now partition

$$\tilde{\Sigma} = \begin{pmatrix} \Sigma_{\mathbf{11}} & \Sigma_{\mathbf{1\tilde{2}}} \\ \Sigma_{\mathbf{\tilde{2}1}} & \Sigma_{\mathbf{\tilde{2}\tilde{2}}} \end{pmatrix}, \quad (2.9)$$

$$\Sigma_{\mathbf{\tilde{2}1}} = \Sigma_{\mathbf{1\tilde{2}}}^T.$$

Define $m \times (n-m)$ and $(n-m) \times m$ matrices

$$D_{\mathbf{21}} = B_{\mathbf{22}}^{-1}B_{\mathbf{21}}, \qquad D_{\mathbf{12}} = D_{\mathbf{21}}^T \quad (2.10)$$

and $(n-m) \times (n-m)$ matrix

$$Q = \Sigma_{\mathbf{11}} - \Sigma_{\mathbf{12}}D_{\mathbf{21}} - D_{\mathbf{12}}\Sigma_{\mathbf{21}} + D_{\mathbf{12}}\Sigma_{\mathbf{22}}D_{\mathbf{21}} = (q_{ij})_{i,j=1}^{n-m}. \quad (2.11)$$

In a similar way to the partition of $\tilde{\Sigma}$, we construct $D_{\mathbf{1\tilde{2}}}$, $D_{\mathbf{\tilde{2}1}}$ as the parts of matrices $D_{\mathbf{12}}$ and $D_{\mathbf{21}}$, respectively, i.e.,

$$D_{\mathbf{1\tilde{2}}} = (d_{ij})_{i=1,,j=n-m+1}^{n-m,n-1}, \qquad D_{\mathbf{\tilde{2}1}} = D_{\mathbf{1\tilde{2}}}^T.$$

Now we generalize Lemma 1 given in Landsman (2008) for the particular riskless case, i.e., when matrix Σ is singular.

Lemma 1. *Let Σ be of the form (2.8) and $\tilde{\Sigma}$ be positive definite, then matrix Q is also positive definite.*

Proof. As Σ is of the form (2.8) the quadratic form $\mathbf{x}^T\Sigma\mathbf{x}$ is, in fact, of the form

$$\mathbf{x}^T\Sigma\mathbf{x} = \tilde{\mathbf{x}}^T\tilde{\Sigma}\tilde{\mathbf{x}},$$

where $\tilde{\mathbf{x}} = (x_1, \ldots, x_{n-1})^T$. For $m > 1$, matrix Q, which was defined in (2.11), can be also written as follows

$$Q = \Sigma_{\mathbf{11}} - \Sigma_{\mathbf{1\tilde{2}}}D_{\mathbf{\tilde{2}1}} - D_{\mathbf{1\tilde{2}}}\Sigma_{\mathbf{\tilde{2}1}} + D_{\mathbf{1\tilde{2}}}\Sigma_{\mathbf{\tilde{2}\tilde{2}}}D_{\mathbf{\tilde{2}1}}. \quad (2.12)$$

As matrix $\tilde{\Sigma} > 0$, it follows from partition (2.9) that $\Sigma_{\mathbf{\tilde{2}\tilde{2}}} > 0$. Consider $(n-1)$ dimensional vector $\tilde{\mathbf{Z}} = (Z_1, \ldots, Z_{n-1})^T = (\mathbf{Z}_1^T, \mathbf{Z}_{\tilde{2}}^T)^T$ distributed $N_n(\mathbf{0}, \tilde{\Sigma})$. Then $\mathbf{Z}_1 = (Z_1, \ldots, Z_{n-m})^T \backsim N_{n-m}(\mathbf{0}_1, \Sigma_{\mathbf{11}})$, $\tilde{\mathbf{Z}}_{\tilde{2}} = (Z_{n-m+1}, \ldots, Z_{n-1})^T \backsim N_m(\mathbf{0}_{\tilde{2}}, \Sigma_{\mathbf{\tilde{2}\tilde{2}}})$ and