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Estimation and inference in the yield curve model with an instantaneous error term

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Abstract

Many variations exist of yield curve modeling based on the exponential components framework, but most do not consider the generating process of the error term. In this paper, we propose a method of yield curve estimation using an instantaneous error term generated with a standard Brownian motion. First, we add an instantaneous error term to Nelson and Siegel's instantaneous forward rate model [C.R. Nelson, A.F. Siegel, Parsimonious modeling of yield curves, Journal of Business 60 (1987) 473–489]. Second, after differencing multiperiod spot rate models transformed using Nelson and Siegel's instantaneous forward rate model [C.R. Nelson, A.F. Siegel, Parsimonious modeling of yield curves, Journal of Business 60 (1987) 473–489], we obtain a model with serially uncorrelated error terms because of independent increment properties of Brownian motion. As the error term in this model is heteroskedastic and not serially correlated, we can apply weighted least squares estimation techniques. That is, this specification of the error term does not lead to incorrect estimation methods. In an empirical analysis, we compare the instantaneous forward rate curves estimated by the proposed method and an existing method. We find that the shape from the proposed estimation equation differ from the latter method when fluctuations in the interest rate data used for the estimation are volatile.

Keywords: Term structure; Exponential components framework; Yield curve; Instantaneous error term; Properties of the error term

1. Introduction

Various theoretical models for term structures have been developed and widely used for the pricing of interest rate derivatives and interest rate risk management. Term structure models are then designed to represent the whole yield curve, including the width and pattern of interest rate fluctuations. From the price theory of derivatives, a nonarbitrary model is proposed in [3,4] and there are the equilibrium models [1,10] that derive bond and option prices. The modeling of the discount function with cubic spline approximation is also suitable for analyzing the term structure or yield curve [5]. Following this model, estimation methods with the Bernstein polynomial, exponential spline, and B-spline are proposed by [7,9,11]. Furthermore, an exponential components model for the instantaneous forward rate curve exists, referred to as the Nelson–Siegel model [6]. We denote this model as the NS model. This model describes the instantaneous forward rate curve in terms of level, slope, and curvature. Several exponential components models that estimate the shape of the yield curve are detailed in [2,8].

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In this paper, we focus on the assumptions regarding the error term introduced in the estimation of the NS model. Despite various extensions to the modeling of the Nelson–Siegel exponential components framework, greater discussion about the specification of the error term is required. For instance, we often estimate the parameters of the NS model using an estimation equation that is a spot rate model with an independently and identically distributed (i.i.d.) error term. Usually, nonlinear least squares estimation or maximum likelihood estimation is applied. In this specification, the error term may not only be heteroskedastic but also serially correlated because the true covariance matrix of the error term is unknown. To eliminate the possibility of misspecification, we propose an estimation equation with a naturally specified error term by adding an instantaneous error term to the instantaneous forward rate model. After differencing the multiperiod spot rate model, it does not suffer from serial correlation of the error term and its statistical properties are clearly represented. As a result, this specification leads to the correct estimation of the NS model. That is, it is only necessary to employ nonlinear weighted least squares estimation.

In the empirical analysis, we compare the estimated parameters, the spot rate, and the instantaneous forward rate curves to those obtained using an estimation equation with a spot rate model with an i.i.d. error term. First, the probability of not rejecting the null hypothesis that the parameter is equal to zero in the proposed estimation equation is much larger than that from the estimation equation with the i.i.d. error term. We confirm that the heteroskedasticity and serial correlation of the error term using the estimation equation with the i.i.d. error term in some sample periods are significant. This implies that the standard errors of the estimated parameters in the estimation equation with the i.i.d. error term are possibly underestimated. Second, the estimated parameters, fitted yield curves, and instantaneous forward rate curves in the proposed estimation equation are similar to those from the estimation equation with the i.i.d. error term in some sample periods. However, the shape of the instantaneous forward rate curves differs when the fluctuations in the interest rate data used in the estimation are more volatile. In this instance, the fitted yield curve of the method proposed is better/worse than the estimation equation with the i.i.d. error term when the number of months to settlement is long/short. Based on these results, we find that the shape of the instantaneous forward rate curves changes depending on the different specification of the error term when interest rate fluctuations are volatile.

The paper proceeds as follows. Section 2 investigates problems in the estimation of the NS model and introduces the correct specification of the error term. Section 3 presents the estimated results of the empirical analysis. The paper concludes in Section 4.

2. Model

In this section, we consider the estimation of an instantaneous forward rate model [6]. Let the instantaneous forward rate at maturity m, which is the period remaining until maturity, be f_m. The NS model is given by:

$$f_m = \beta_0 + \beta_1 \exp\left(\frac{-m}{\tau}\right) + \beta_2 \left(\frac{m}{\tau}\right) \exp\left(\frac{-m}{\tau}\right), \tag{1}$$

where β_0, β_1, β_2, and τ are parameters to be estimated from the interest rate data. The instantaneous forward rate function is approximated by the sum of the constant term and two exponential functions. As m approaches infinity and zero, the value of f_m becomes β_0 and $\beta_0 + \beta_1$, representing a consol bond and the instantaneous spot rate, respectively. That is, the first and second terms represent the contributions of the long- and short-term components to the forward rate curve. β_1 takes a negative value when the shape of the forward rate curve is upward sloping. When it is a reverse yield, β_1 takes a positive value. β_2 is positive and negative when the medium-term component on the forward rate curve is hump and U-shaped, respectively. τ controls the exponential convergence speeds of the second and third terms. The large value of τ creates a gentle slope and slows down the convergence speed to the shape of the forward rate curve in the long run. Some authors [2,8] propose various models based on the NS model through the addition of another exponential term and the different convergence speeds of the second and third terms. We do not consider these models in this paper as we focus on the estimation methods of the NS model.

Using the instantaneous forward rate f_m, we can express the spot rate r_m as:

$$r_m = \frac{1}{m}\int_0^m f_s ds = \beta_0 + (\beta_1 + \beta_2)\frac{[1 - \exp(-m/\tau)]}{(m/\tau)} - \beta_2 \ \exp\left(\frac{-m}{\tau}\right). \tag{2}$$

The spot rate is derived by integrating the instantaneous forward rates from zero to m and dividing by m. Traditionally, after adding an error term to the spot rate representation we apply nonlinear least squares (NLS) or maximum likelihood