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A numerical approach to obtain the yield curves with different risk-neutral drifts

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ABSTRACT

In this paper we consider the possible dependence of the market price of risk on time and interest rates. This fact gives as a result that the risk-neutral drift, which is one of the coefficients of the pricing equation, also depends on time and interest rates. Then, we estimate the risk-neutral drift directly from the slope of the yield curve. This approach is very accurate as we show with a numerical experiment. In order to obtain the term structure we also propose a suitable finite difference method, which converges to the true solution. Finally, we obtain and compare the yield curves with data from the US Treasury Bill market.

1. Introduction

The term structure of interest rates has fascinated generations of researchers, [1–4]. This fact should not come as a surprise. An understanding of the stochastic behavior of interest rates is important for the conduct of the monetary policy, the public debt management, the expectations of the real economy activity and inflation, the risk management of a portfolio of securities, and the valuation of interest rate derivatives, [5].

This paper focuses attention on one-factor short rate models. Although they have several shortcomings, they are still very attractive for academics and practitioners. They promise to offer stable and consistent models, with parsimonious structure for the fundamental behavior of interest rates and term structure.

In the empirical implementation of the one-factor models there is only one state variable, the instantaneous interest rate. However when we use the Theory of Arbitrage we also need to use the risk-neutral probability or equivalently, the market price of risk which is unobservable. Traditionally, this function has been considered arbitrary and even constant to find a closed-form solution. However, this fact can lead to misspecification.

The aim of this paper is to analyze the effect on the market price of risk dependence on time and interest rate on the yield curves. The cost of considering more realistic functions in the model is that a closed-form solution is not known. However this is not a problem because we propose an efficient numerical method to provide an accurate approximated solution for the term structure problems. Moreover, we propose to estimate the short rate risk-neutral drift directly from the slope of the yield curve. Therefore, we have to estimate neither the interest rate drift nor the market price of risk. This fact reduces the misspecification of these functions, the computational cost of the model and finally, the term structure errors.

The rest of the paper is organized as follows. Section 2 briefly describes a term structure model with one stochastic variable, the instantaneous interest rate. Then, we show some of the most well-known models in the term structure literature, [2,6]. Moreover we propose some generalizations for the market price of risk of these models. As a consequence,

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we obtain some new models that we call MODCIR and MODCKLS, respectively. In Section 3 we show an efficient approach to estimate the risk-neutral drift directly from data in the markets. Furthermore, we make a numerical experiment to show the properties of this approach. In Section 4 we propose a finite difference method to solve the pricing equation when a closed-form solution is not known and we show some properties as its convergence. In Section 5, we estimate the parameters of traditional [2] and [6], and new MODCIR and MODCKLS models from recent US Treasury Bill data and we obtain the yield curves. We conclude in Section 6.

2. The model

In this study, we focus on a Markov model with only one state variable, the instantaneous interest rate, r, which follows a stochastic process of the type $\mathrm{d}r_t = \mu(r_t)\mathrm{d}t + \sigma(r_t)\mathrm{d}z_t$, where z_t is the standard Wiener process, $\mu(r_t)$ is the instantaneous drift and $\sigma(r_t)$ is the instantaneous volatility. Let $P(t, r; T)$ denote the price at time t of a zero-coupon bond maturing at time T, with $t \leq T$. The bond is assumed to have a maturity value of one unit, i.e.

$$P(T, r; T) = 1. \tag{1}$$

As the short rate is assumed to follow a diffusion dynamics, Itô's lemma implies that zero-coupon bonds follow diffusion dynamics as well. Therefore, by means of the risk-neutral pricing approach a zero-coupon bond price, when scaled by the money market account, is a martingale under a specific, artificially constructed probability measure Q, [5]. Therefore:

$$P(t, r; T) = E_t^{Q}\left[\exp\left(-\int_t^T r_u \mathrm{d}u\right) P(T, r; T)\right], \tag{2}$$

where $\mathrm{d}r_t = (\mu(r_t) - \sigma(r_t)\lambda(t, r_t))\mathrm{d}t + \sigma(r_t)\mathrm{d}z_t$, is the interest rate under Q or the risk-neutral interest rate. Here $\lambda(t, r)$ is the market price of risk and satisfies the so-called Novikov condition, [7]. Furthermore by means of the Feynman–Kac theorem, [8], it can be easily proved that (2) is the solution of the following partial differential equation:

$$\frac{\partial P}{\partial t} + (\mu(r) - \sigma(r)\lambda(t, r))\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2(r)\frac{\partial^2 P}{\partial r^2} - rP = 0 \tag{3}$$

which is the same for all interest rate derivatives, [7]. In order to obtain the term structure, μ, σ, and λ must be estimated. Then, these functions are replaced in the pricing equation (3) and it is solved by taking into account the corresponding final condition (1).

Finally, the yield to maturity, $R(t, r; T)$, is the internal rate of return at time t on a zero-coupon bond with maturity date $\tau = T - t$, given by

$$R(t, r; T) = -\frac{1}{\tau}\log P(t, r; T), \quad t \leq T. \tag{4}$$

In the literature of the term structure, there are different models depending on the different assumptions about the interest rates and the market price of risk. From all of them, the traditional models of [1,2] and [6] are the most well known in the literature. In this paper, we focus on CIR and CKLS models proposed by [2] and [6], respectively. In the CIR model, the interest rate is driven by the following stochastic process $\mathrm{d}r_t = \beta(m - r_t)\mathrm{d}t + \sigma\sqrt{r_t}\mathrm{d}z_t$ and the market price of risk is as follows $\lambda(r) = \frac{\lambda}{\sigma}\sqrt{r}$, $\beta, m, \sigma, \lambda \in \mathbb{R}$. These choices of $\mu(r)$, $\sigma(r)$ and $\lambda(t, r)$ functions provide an affine model and its solution is as follows $P(t, r; T) = P(r, \tau) = \exp(A(\tau) + B(\tau)r)$. $A(\tau)$ and $B(\tau)$ are widely well known and extensively analyzed in the literature, for example [2].

In [6] the volatility is allowed to be of the form $\sigma(r) = \sigma_0 r^\gamma$, with $\sigma_0,\ \gamma \in \mathbb{R}$. Therefore they do not make any assumption about the market price of risk. However, in the literature it has been considered as a constant for simplicity. In this paper, we will also refer to the CKLS model when the market price of risk function is constant. In contrast to the CIR model, the CKLS model is not affine and a closed-form solution is not known. Then we have to deal directly with the pricing equation (3) to obtain an approximate solution.

One of the goals of this paper consists of considering the possible dependence of the market price of risk on time and interest rates. Thus, we propose the following functions: $\lambda(t, r) = (\lambda_1 + \lambda_2 t)\sqrt{r}$ in the CIR model; and $\lambda(t, r) = (\lambda_1 + \lambda_2 t)r^{1-\gamma}$ when $0 \leq \gamma < 1$ in the CKLS model. These assumptions give as a result two models with the same risk-neutral drift: $g(t, r) = \alpha_1 + (\alpha_2 + \alpha_3 t)r$, with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, but different volatilities, which we call MODCIR and MODCKLS, respectively.

Following [9], MODCIR is a non-homogeneous affine model and the solution is as follows $P(t, r; T) = \exp(A(t, T) + B(t, T)r)$. If we replace this solution and its derivatives in (3) we obtain the following system of ordinary differential equations:

$$A'(t) = \beta m B(t),$$

$$B'(t) = [\beta + \sigma(\lambda_1 + \lambda_2 t)]B(t) + \frac{1}{2}\sigma^2 B^2(t) - 1, \quad 0 < t < T,$$

with final conditions: $A(T) = 0$, $B(T) = 0$, which is far easier and less time consuming to solve than a partial differential equation. Although the solution of this system of ordinary differential equations is unknown it is relatively easy to solve using numerical methods.