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A note on scale functions and the time value of ruin for Lévy insurance risk processes

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ABSTRACT

We examine discounted penalties at ruin for surplus dynamics driven by a general spectrally negative Lévy process; the natural class of stochastic processes which contains many examples of risk processes which have already been considered in the existing literature. Following from the important contributions of [Zhou, X., 2005. On a classical risk model with a constant dividend barrier. North Am. Act. J. 95–108] we provide an explicit characterization of a generalized version of the Gerber–Shiu function in terms of scale functions, streamlining and extending results available in the literature.

1. Introduction

Originally motivated by the pricing of American claims, Gerber and Shiu (1997, 1998) introduced in risk theory a function that jointly penalizes the present value of the time of ruin, the surplus before ruin and the deficit after ruin for Cramér–Lundberg-type processes. This expected discounted penalty, by now known as the Gerber–Shiu function, has been frequently and recursively studied in settings of increasing generality as well as being the named theme of two international workshops in 2006 and 2008. Although far from exhaustive on account of the sheer volume of relevant literature, a list of key papers which pertains to generalizations of the Cramér–Lundberg process includes for example Dickson (1992, 1993), Gerber and Shiu (1997, 1998), Gerber and Landry (1998), Lin and Willmot (1999), Yang and Zhang (2001), Cai and Dickson (2002), Tsai and Willmot (2002), Cai (2004), Garrido and Morales (2006), Morales (2007) and Morales and Olivares (2008). The general setting which fits all of these papers is to model the risk process as having *stationary and independent increments with no positive jumps*. Excluding of course the undesirable case of monotone decreasing paths, the latter class is commonly referred to as spectrally negative Lévy processes. In the current actuarial setting we refer to them as *Lévy insurance risk processes*.

A common feature of the existing literature is to reduce the analysis of Gerber–Shiu functions to the study of integro-differential equations and/or Volterra equations. In the case of a compound Poisson jump structure the nature of these equations boils down to conditioning on the first jump and considering the recursive nature of the Gerber–Shiu function.

Whilst intuitively appealing, these approaches can be argued to suffer from some limitations as far as dealing with Lévy risk insurance processes. For example, the integro-differential equation can itself only be worked with under the assumption that there is sufficient smoothness in the Gerber–Shiu function, which is *a priori* a highly non-trivial fact to establish. The associated Volterra equation (see Section 4 for further details) quickly becomes very involved, with different components of the risk process (e.g., bounded and unbounded variation, or continuous and discontinuous paths) requiring separate consideration. Existing calculations show that it is a lengthy procedure to obtain a Volterra equation for the Gerber–Shiu function in the case of infinite activity Lévy insurance risk processes (i.e., countably infinite negative jumps in bounded intervals of time) as this is typically done via compound Poisson approximation. Another difficulty for Volterra/integro-differential equations is that, apart from a handful of exceptions, there is no general theory which offers an identifiable solution, even in the simplest setting of a general compound Poisson jump structure. One such exceptional example would be the classical case of claims whose jump distribution has a Laplace transform which is rational (c.f.: Borovkov, 1976) which includes for example the phase-type distributions.

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Zhou (2005) makes an important contribution from the general point of view of Lévy insurance risk processes and introduces the use of so-called *scale functions* in his analysis of the Gerber–Shiu function. Although his computations are restricted to the case of a compound Poisson jump structure, the analysis still applies verbatim for the case of a general spectrally negative Lévy process.

In this paper, we build on Zhou's results and advocate further the virtues of characterizing expected discounted penalties through the use of scale functions and fluctuation theory of Lévy processes when the surplus is driven by a spectrally negative Lévy process. For a modern overview of the theory of scale functions, see for example Bertoin (1996) and Kyprianou (2006). In these two books one also finds further historical references regarding their use in the work of Takács and Zolotarev in the 1960s, and the work of the Kiev school led by Korolyuk (using what they called the method of resolvents) from the 1970s.

As the forthcoming sample computations will hopefully demonstrate, working with scale functions allows one to deal with Lévy insurance risk processes, without having to disentangle, for example, a perturbation from a claims component. The approach equally applies to risk processes characterized by components of bounded and unbounded variation, finite and infinite activity as well as by continuous and discontinuous paths, providing in turn solutions that only require inversion of a single Laplace transform.

From the computational point of view, the scale function approach can rely on the vast literature of Laplace transform methods, which has grown extensively over the last few years, and has found numerous applications in option pricing (e.g.: Duffie et al., 2000; Lee, 2004) and integro-differential equations (e.g.: Kythe and Puri, 2002; Babolian and Shamloo, 2008). Rogers (2000) and Surya (2008) have provided robust methods for numerically computing scale functions. Until recently there were only a handful of explicitly known examples of scale functions (this fact is also reflected by the small variety of concrete examples of the Gerber–Shiu function). Recent work however has produced many completely explicit examples of scale functions (e.g.: Hubalek and Kyprianou, 2008; Patie, 2008; Chaumont et al., 2009) and in particular Hubalek and Kyprianou (2008); Kyprianou and Rivero (2008) and Patie and Kyprianou (unpublished manuscript) outline a method from which many more examples can be computed than the articles themselves had the space for.

Other advantages of scale functions can be seen in recent literature that looks at dividend payments (for example in the form of reflection or refraction strategies) and have proved to be key in understanding optimal barrier strategies (see: Zhou, 2005; Renaud and Zhou, 2007; Kyprianou and Palmowski, 2007; Albrecher et al., 2008; Kyprianou and Zhou, 2009; Loeffen, 2009; Kyprianou et al., 2008b; Kyprianou and Loeffen, 2008; Loeffen, 2008a,b). In the context of last ruin times Chiu and Yin (2005) and Baurdoux (in press) also make extensive use of the theory of scale functions. It can be argued that the role and functional robustness of the scale function across this broad range of topics provides a unified reading. Moreover, there is a setting to which the use of more general analogues of scale functions pertains; that is the first passage problem of a general Lévy process and similar such processes such as Markov additive processes and positive self-similar processes. Here one also finds a growing body of work (e.g.: Doney and Kyprianou, 2006; Klüppelberg and Erder, 2008; Breuer, in press; Kyprianou et al., 2008a; Chaumont et al., 2009; Caballero and Chaumont, 2006; Chen and Sheu, 2009).

On the downside, scale functions are arguably a particular phenomenon of risk processes with stationary and independent increments and negative jumps. For example, models of risk which depart from the Poissonian jump structure or introduce path dependencies (and therefore destroy the convenience of stationary independent increments) do not necessarily have a comfortable analogue of the theory of scale functions. Moving outside the class of Lévy insurance risk processes however is not completely hopeless as far as scale functions are concerned. For example, the recent paper of Gerber et al. (2006) shows that a more general notion of the scale function exists for risk processes which are modelled as strong Markov processes with stationary increments and no positive jumps. Whilst it is not clear exactly how far one may push the theory of scale functions, it is worthy of note that in cases where the use of scale functions fail, the use of Volterra/integro-differential equations is often still applicable.

The remainder of this note is organized as follows: In Section 2, we look in more detail at the decomposition of a spectrally negative Lévy process in relation to existing models and consider a general class of expected discounted penalty functions (as introduced in Biffis and Morales, 2008) for which we shall exemplify the use of scale functions with some sample computations collected in the form of a main theorem and proof. In Section 3 we give concrete examples making use of recent explicit examples of scale functions. Section 4 offers some concluding remarks.

2. Lévy insurance risk processes and discounted penalties

Recall that the Cramér–Lundberg model corresponds to a Lévy process $X = \{X_t : t \geq 0\}$ with law $\mathbb{P}$ and characteristic exponent given by

$$\Psi(\theta) = -\log \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x}\mathbb{P}(X_1 \in \mathrm{d}x) = -\mathrm{i}c\theta + \lambda \int_{(0,\infty)} \left(1 - \mathrm{e}^{-\mathrm{i}\theta x}\right) F(\mathrm{d}x),$$

for $\theta \in \mathbb{R}$ such that $\lim_{t\uparrow\infty} X_t = \infty$. In other words, X is a compound Poisson process with arrival rate $\lambda > 0$ and negative jumps, corresponding to claims, having common distribution function F with finite mean $1/\mu$ as well as a drift $c > 0$, corresponding to a steady income due to premiums, which necessarily satisfies $c - \lambda/\mu > 0$; the *safety loading condition*.

As mentioned in the Introduction, we work with a general spectrally negative Lévy process $X = \{X_t : t \geq 0\}$ and refer to it as a *Lévy insurance risk process*. The analogous condition to the safety loading condition in this general setting (that is to say, the necessary and sufficient condition that ensures $\lim_{t\uparrow\infty} X_t = \infty$) is $\mathbb{E}(X_1) \in (0,\infty)$. (Note that it is impossible for the latter expectation to equal $+\infty$, cf. Chapter VII of Bertoin, 1996, or Chapter 8 of Kyprianou, 2006). Under the latter assumption the Lévy–Khintchine formula for the characteristic exponent of a Lévy insurance risk process may always be written in the form

$$\Psi(\theta) = -\log \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\theta x}\mathbb{P}(X_1 \in \mathrm{d}x) = -\mathrm{i}\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{(0,\infty)} (1 - \mathrm{e}^{-\mathrm{i}\theta x} - \mathrm{i}\theta x)\Pi(\mathrm{d}x) \tag{1}$$

for $\theta \in \mathbb{R}$, where $a > 0$, $\sigma \geq 0$, the Lévy measure, Π, is concentrated on $(0,\infty)$ and necessarily satisfies $\int_{(0,\infty)} (x \wedge x^2)\Pi(\mathrm{d}x) < \infty$. Note that, although jumps are negative, we have made the unusual arrangement in the Lévy–Khintchine formula of supporting the Lévy measure Π on the positive half line. Moreover, not using a truncation function against the linear term in the integrand of Ψ, restricting $a > 0$ and allowing integrability against Π of $(x \wedge x^2)$ as opposed to the usual $(1 \wedge x^2)$ are all adjustments to the usual form of the Lévy–Khintchine formula which are possible thanks to the safety loading condition $\mathbb{E}(X_1) \in (0,\infty)$.

Different authors interpret the different parts of the characteristic exponent, and the associated Lévy–Itô decomposition, in different ways. The case that $\Pi(0,\infty) < \infty$ corresponds to