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Generalized EOQ formula using a new parameter: Coefficient of backorder attractiveness

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ABSTRACT

This paper introduces and examines a generalized EOQ formula, based on the model with linear and fixed backordering costs. The new square-root formula is a combination of two well-known classical models, the basic EOQ model without stockouts and the EOQ model with backorders and linear backordering costs. Helping to combine the two is a new parameter, a fractional coefficient capturing the attractiveness of backorders. The coefficient is explained and discussed. The paper concludes with a brief discussion of generalizations in EOQ models, including EPQ.

1. Introduction

This paper introduces and examines a possible generalization of the widely-known classical EOQ formula. It is well known that very extensive work has been done over the years on several variations of EOQ under several different assumptions. Since it is not the intention of this paper to do a large literature review, we will limit our citations mostly to those that were immediate predecessors leading to this paper. A relatively recent stream of research is the utilization of algebraic methods instead of calculus to analyze inventory models. Examples of that approach include Grubbstrom (1995), Grubbstrom and Erdem (1999), Cárdenas-Barrón (2001, 2010), Huang (2003), Ronald et al. (2004), Chang et al. (2005), Sphicas (2006), Minner (2007), Leung (2008). The above selected references employed mostly algebraic methods, but that should not give the impression that calculus is no longer used in the inventory literature. Some recent examples of published papers based on calculus are Yang (2007), Leung (2008), Pentico and Drake (2009), and Chung and Cárdenas-Barrón (2012). The work reported here is based on the model with linear and fixed backordering costs and is specifically motivated by the properties and results first presented in Sphicas (2006). In that paper the solution was analyzed using only algebra. Later, the same model was examined and similar results were obtained using calculus in Chung and Cárdenas-Barrón (2012), and also using a combination of analytic geometry and algebra in Cárdenas-Barrón (2011). In Section 2 the main results from Sphicas (2006) are reviewed, since they form the basis of the new development proposed here. In Section 3 the modified square-root formula and generalized EOQ is developed. Introduced and defined is a new parameter, a fractional coefficient capturing the attractiveness of backorders. The logic of that and the interpretation and significance of the coefficient are discussed in Section 4. Finally in Section 5 some discussion of generalizations in EOQ models is presented, including the case of EPQ.

2. The EOQ model with backorders and two backordering costs

Since we will be making frequent reference to several versions of the EOQ, we will employ the following convention: EOQ_0 will refer to the classical no-stockouts model, and the corresponding famous Harris (1913) formula will be called $EOQ_0 = \sqrt{2KD/h}$. Similarly, EOQ_1 will refer to the basic backordering model with only a linear backordering cost, and the optimal quantity of that will be called $EOQ_1 = \sqrt{(2KD/h)(1+h/p)}$. This paper is motivated by some of the conclusions on the EOQ with two backordering costs, fixed and linear. To avoid having to refer to this model by a long explanation, let us define it as EOQ_2. Thus EOQ_2 refers to the problem of finding values for the parameters Q and S (subject to $0 \le S \le Q$), which minimize the total cost

$$TC = \frac{KD}{Q} + \frac{h(Q-S)^2}{2Q} + \frac{pS^2}{2Q} + \frac{\pi DS}{Q}$$

where K, D and h are the usual parameters for ordering cost, demand per unit of time, and holding per unit per unit of time. All customers are willing to wait for late delivery, and backordering results in two costs, a linear cost per unit per unit of time p, and a fixed backordering cost per unit π. We mostly refer to Sphicas (2006), where the model was analyzed using algebra. This model had been examined much earlier, by Johnson and Montgomery

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(1974), and analyzed using calculus, but their analysis was incomplete. Subsequent to Sphicas (2006), at least two more recent papers (Cárdenas-Barrón (2011); Chung and Cárdenas-Barrón (2012)), examined the same model and obtained similar results using different approaches. It does not matter how the results were originally proved, especially since there is no difference in any of the results. Here we start with some of the major conclusions in that Sphicas (2006) paper.

Result 1: The optimal solution of EOQ_2 is a conditional one. There are two cases. In Case 1, when $\pi D \geq \sqrt{2KDh}$, backorders are too expensive, and none should be used. In this case the optimal policy is to use the basic $EOQ_0 = \sqrt{2KD/h}$.

Result 2: In Case 2, when $\pi D \leq \sqrt{2KDh}$, backorders are attractive and should be used to minimize total cost. In this case the optimal solution of EOQ_2 is uniquely given by the order size

$$Q^* = \sqrt{\frac{2KD(h+p) - \pi^2 D^2}{hp}} \tag{1}$$

Result 3: For Case 2 (backorders attractive), the corresponding level of backorders is $S^* = (hQ^* - \pi D)/(h+p)$, the maximum inventory level $Q^* - S^* = (pQ^* + \pi D)/(h+p)$, and the minimum total Cost $TC^* = h(Q^* - S^*)$. All values are real and valid as long as the condition for this case ($\pi D \leq \sqrt{2KDh}$) is satisfied.

Result 4: The size of Q^* given in (1) is in-between the basic $EOQ_0 = \sqrt{2KD/h}$ and $EOQ_1 = \sqrt{(2KD)/h(1+(h/p))}$. The optimal total cost of EOQ_2 is in-between the corresponding $TC_0{}^* = \sqrt{2KDh}$ and $TC_1{}^* = \sqrt{2KDh\{h/(h+p)\}}$. (It may be noted that in Sphicas (2006) some graphs are included illustrating the various relative sizes).

Result 5: As functions of π (in the appropriate range of Case 2 values of π), Q^* and S^* are decreasing, while $Q^* - S^*$ and TC^* are increasing

3. New formula

The fact that EOQ_2 has results that place it between EOQ_0 and EOQ_1 is the main motivation for this work. We are attempting to express the results in a form that would explicitly reflect that relationship. There are two difficulties that need to be overcome: The fact that the solution of EOQ_2 is a conditional one, and the nature of the square root formula (1) which appears at first to be rather unwieldy. It turns out that both difficulties can be overcome by introducing a fractional value, which is developed in steps.

A starting point is to work with the conditions determining the two distinct cases and use a ratio. Since the two cases are defined by $\pi D \geq \sqrt{2KDh}$ for Case 1, and $\pi D \leq \sqrt{2KDh}$ for Case 2, a natural ratio to consider is $\alpha = \pi D/\sqrt{2KDh}$. Then the two conditions become $\alpha \geq 1$ for Case 1 and $\alpha < 1$ for Case 2. Next we note that α cannot be negative, hence we can square, and substitute $\pi^2 D^2 = \alpha^2 2KDh$ in (1). After a bit of algebra, the result becomes

$$Q^* = \sqrt{\frac{2KD}{h}\left(1 + (1-\alpha^2)\frac{h}{p}\right)} \tag{2}$$

We can now restate the conditional solution of the EOQ_2 is follows: In Case 1 ($\alpha \geq 1$) the optimal value of Q^* is $EOQ_0 = \sqrt{2KD/h}$. In Case 2 ($\alpha < 1$) the optimal value of Q^* is correctly given by (2). One logical step remains. Noting that EOQ_0 is exactly the same as the right hand side of (2) with $\alpha = 1$,we would like to make $\alpha = 1$ true for Case 1. Mathematically it might be acceptable to specify something like "if $\alpha \geq 1$ then set $\alpha = 1$". But instead of that, we can accomplish the same result by introducing another parameter β, conditionally defined as $\beta = 1 - \alpha^2$ if $\alpha < 1$ and $\beta = 0$ if $\alpha \geq 1$. Equivalently, we can state the definition as $\beta = \mathrm{Max}(0, 1 - \alpha^2)$, i.e.,

$$\beta = \mathrm{Max}(0, 1 - \pi^2 D/2Kh) \tag{3}$$

Then we get the main result of this paper:

Proposition 1. With a fraction β in (0,1) defined as $\beta = \mathrm{Max}(0, 1 - \pi^2 D/2Kh)$, Model EOQ_2 has a unique solution with optimal order size Q^* given by

$$EOQ_\beta = \sqrt{\frac{2KD}{h}\left(1 + \beta\frac{h}{p}\right)} \tag{4}$$

This is proposed as a possible generalization of the classical EOQ models. Several comments on the definition and interpretation of the coefficient β will be given later. First, it is useful to confirm that (4) indeed includes both cases of EOQ_2. In Case 1, (backorders too expensive), when $\pi D \geq \sqrt{2KDh}$, the definition of β ensures that $\beta = 0$, and (4) reduces to $EOQ_0 = \sqrt{2KD/h}$. In Case 2, (backorders attractive), the value of β is a fraction in (0,1), and when $\beta = 1 - \pi^2 D/2Kh$ is substituted in (4), the result is exactly (1). Thus (4) applies to both Case 1 and Case 2, and we can use that as a general solution without having to consider distinct cases. The corresponding values of S^*, $Q^* - S^*$, and TC^*, mentioned earlier in Results 3, 4 and 5, can also be rewritten in terms of the new coefficient β, as the next proposition states. All expressions for the decision variables are shown to depend only on $\sqrt{2KD/h}$ (the EOQ_0) and only two parameters, β and r. The parameter r is introduced for convenience in the formulas, and is the ratio of two unit costs, $r = h/p$. It may be noted that this ratio is not necessarily a fraction. Perhaps one might examine different scenarios where the unit cost per unit of time for holding may be more or less expensive than backordering, but here we take r to have any positive value.

Proposition 2. For EOQ_2, the optimal values of Q^*, S^*, $Q^* - S^*$ and TC^* are given by:

$$Q^* = \sqrt{2KD/h}\sqrt{1+\beta r}$$

$$S^* = \sqrt{2KD/h}\, r\,(\sqrt{1+\beta r} - \sqrt{1-\beta})/(1+r)$$

$$Q^* - S^* = \sqrt{2KD/h}\,(\sqrt{1+\beta r} + r\sqrt{1-\beta})/(1+r)$$

$$TC^* = \sqrt{2KD/h}\,(\sqrt{1+\beta r} + r\sqrt{1-\beta})/(1+r)$$

Some of the formulas in Result 3 of the previous section involved πD. That can be safely substituted by $\pi D = \sqrt{2KDh}\,\sqrt{1-\beta}$, since the conditional definition of the fraction β has no effect on the results. Some of the formulas need some algebraic manipulation but should be easy to verify. It may be noted that all values are nonnegative. The only term with a negative sign appears in S^*, but $\sqrt{1+\beta r}$ is ≥ 1 while $\sqrt{1-\beta}$ is ≤ 1, so that is not a concern.

Proposition 3. As shown explicitly in Proposition 2, all values for the decision variables can be scaled in terms of the EOQ_0 value, which itself does not need to be known in advance. Thus, knowing that the optimal Q size for this model is simply EOQ_0 multiplied by $\sqrt{1+\beta r}$, or that the optimal S size is EOQ_0 multiplied by $r(\sqrt{1+\beta r} - \sqrt{1-\beta})/(1+r)$ is more general than specific numerical values. Furthermore, the actual number of distinct parameter values needed to completely solve the model is basically reduced to three: The value EOQ_0, the fractional value of β, and the ratio of costs r. All other parameter values, such as D, K, h, p and π are indirectly included in these three.

The fraction $f = (\sqrt{1+\beta r} + r\sqrt{1-\beta})/(1+r)$ may be of particular interest, since it shows the ratio of costs, $f = TC^*/TC_0{}^*$. Thus $1 - f$ would be the extent of the cost savings resulting from backordering